Static and infalling quasilocal energy of charged and naked black holes

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Abstract

We extend the quasilocal formalism of Brown and York to include electromagnetic and dilaton fields and also allow for spatial boundaries that are not orthogonal to the foliation of the spacetime. The extension allows us to study the quasilocal energy measured by observers who are moving around in a spacetime. We show that the quasilocal energy transforms with respect to boosts by Lorentz-type transformation laws. The resulting formalism can be used to study spacetimes containing electric or magnetic charge but not both, a restriction inherent in the formalism. The gauge dependence of the quasilocal energy is discussed. We use the thin shell formalism of Israel to reinterpret the quasilocal energy from an operational point of view and examine the implications for the recently proposed AdS/CFT inspired intrinsic reference terms. The distribution of energy around Reissner-Nordström and naked black holes is investigated as measured by both static and infalling observers. We see that this proposed distribution matches a Newtonian intuition in the appropriate limit. Finally the study of naked black holes reveals an alternate characterization of this class of spacetimes in terms of the quasilocal energies.

1 Introduction

Gravitational thermodynamics continues to be one of the most active areas of research in gravitational physics. The interest began in the early 1970's with Bekenstein's recognition that if the temperature of a black hole is proportional to its surface gravity and its entropy is proportional to the surface area of its horizon, then the laws of black hole mechanics are laws of thermodynamics [1]. These speculations were confirmed by Hawking's discovery that a black hole emits radiation as a perfect black body with temperature proportional to its surface gravity [2] and by calculations based on Gibbons and Hawking's proposal of the Euclidean path integral formulation of gravity[3] which predicted that a black hole has an entropy equal to one quarter of its surface area.

In the quest for a theory of quantum gravity, these laws of black hole thermodynamics are amongst the few solid results available. As such the investigation of their features and the search

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for a statistical mechanics underlying them have been areas of tremendous activity and interest ever since. Indeed, the entropy/area relationship is viewed as an acid test which all aspiring quantum gravity theories must pass. In particular, in recent years there have been successful attempts to show that both string theory [4] and canonical quantum gravity [5] correctly predict the entropy of black holes.

On a more prosaic level however, one of the most popular current approaches to semi-classical gravitational thermodynamics is based on the Euclidean path integral formalism and a Hamiltonian analysis of the action functional that was proposed by Brown and York [6, 7]. They considered a spatially bounded region of spacetime and defined a stress energy tensor over the history of that system's boundary as the functional derivative of the action with respect to the three metric on that boundary. Foliating the spacetime they then defined quasilocal energy (QLE), angular momentum, and spatial stress densities as projections of the stress energy into the corresponding foliation of the boundary. Reference terms subtracted from these densities are defined by embedding that same boundary into a reference spacetime and recalculating the quasilocal quantities for that new embedding. Integrating the densities over one leaf of the foliation of the boundary, the full quasilocal quantities (rather than just tensor densities) are defined for that "instant" of time.

With the energy, momentum, and pressure defined on the boundary one can do gravitational thermodynamics (see as examples [7, 8, 9]), study quantum mechanical black hole pair creation [10, 11, 12], and investigate the distribution of gravitational energy in a variety of spacetimes [13, 14] – a matter of interest in its own right. It has been shown that in the appropriate limits, the QLE agrees with many of the usual definitions of the total energy of the system. For example in asymptotically flat spacetime it has been shown to be equivalent to the Arnowitt-Deser-Misner (ADM) energy [15] (in [6]) and the Trautman-Bondi-Sachs (TBS) [16] energy (in [17]). In asymptotically anti-de Sitter spacetimes it has been shown [8, 18] to be equivalent to the Abbot-Deser energy [19]. Very recently interest in this subject has been renewed with the AdS/CFT inspired redefinition of the reference terms with respect to intrinsic rather than extrinsic curvatures. The formalism can then be used to investigate the thermodynamics of a new group of (mainly AdS) spacetimes [20] that were inaccessible to the embedding reference term approach.

Within the original quasilocal formalism however, there was an acknowledged incompleteness. Namely, it was assumed that the spatial boundary was always orthogonal to the spacetime foliation. This simplified some calculations and was true for all of the standard examples where one considers static spherically symmetric boundaries in static spherically symmetric spacetimes foliated according to the usual time coordinate. From a theoretical standpoint this assumption restricted the variations used to define the stress tensor (and calculate the equations of motion) to those that preserve the orthogonality of the foliation and the boundary. Practically, it meant that it was extremely difficult to calculate quasilocal quantities measured by non-static observers. As an example, the history of a spherical set of observers falling into a Schwarzschild black hole would not be orthogonal to the foliation defined by the usual time coordinate t.

Only a few papers have considered the consequences of dropping the orthogonality restriction. Hayward [21] modified the gravitational action so that its variation is well defined for arbitrary variations of the metric. Lau partially dealt with the issue in [22] though his main concern was the definition of quasilocal quantities with respect to Ashtekar variables. Hawking and Hunter developed a non-orthogonal formulation which explicitly depended on the angle of intersection between the foliation and the boundary [23]. This dependence could only be removed by carefully foliating the reference spacetime to have the same intersection angles at the boundary, making the calculations quite cumbersome.

Our own considerations on this issue were the subject of a recent paper [24]. In contrast with the approach of Hawking and Hunter we focused on the foliation of the spatial boundary rather than that of the spacetime as a whole. With this approach the formalism was independent of the foliation of the rest of the space-time even without the inclusion of reference terms. Computationally it was easier to apply and we also argued that our approach was the more natural one to take (see [24] or section 3 of this paper for more details). In our approach it is relatively easy to calculate such quantities as the quasilocal energy measured by observers falling into a black hole.

An interesting class of black holes, almost tailor-made to be investigated by our methods, has recently been discussed by Horowitz and Ross [25]. These so-called naked black holes are massive near-extreme static black hole solutions to the equations of string theory in the low energy limit. Observers who aren't moving with respect to such holes and who are just outside the event horizon measure only small curvature invariants. However, observers falling into the holes along geodesics see Planck scale curvatures and experience correspondingly massive crushing tidal forces as they approach the horizon. The holes were dubbed "naked" because Planck scale curvatures can be observed outside of their event horizons.

In this paper we extend our previous work on non-orthogonal boundaries (reviewed in section 3.1) to include appropriate matter fields (section 3.2) so that we can calculate the QLE measured by these two sets of observers and compare it to the curvatures that they observe. The appropriate matter fields in this case are an electromagnetic field coupled to a dilaton field. As such along the way we will examine the gauge dependence of the quasilocal energy, see why this gauge dependence arises, and investigate exactly how it manifests itself. We shall see that the quasilocal energy naturally breaks up into a gauge independent geometric part and a gauge dependent part.

The naked black holes that we wish to study are magnetically charged so we necessarily also confront issues related to electromagnetic duality (section 3.3). We will show that the formalism that we have set up does exhibit this duality, but at the same time is not suited to dealing with dyonic black holes. In its current formulation, it can handle situations with electric or magnetic charge but not both simultaneously.

Having dealt with these gauge-theoretic issues which are essentially unrelated to our non-orthogonal extension we then examine in some detail how quasilocal quantities transform with the motion of sets of observers who are measuring them (section 3.4). We significantly improve our earlier treatment [24] and see that the quasilocal energy in particular transforms according to simple Lorentzian-type laws (though in this case there are two different velocities involved in those laws).

Then, before turning to the examples, we examine the very close correspondence between the quasilocal formalism and the work by Israel explaining the behaviour of thin shells of matter in general relativity (section 3.5). This correspondence provides support for the quasilocal notion of energy and enables us to reinterpret it in an operational way. In the light of this reinterpretation we then examine recent AdS/CFT inspired work [20] on reference terms for the quasilocal energy.

The first spacetimes that we investigate in the light of the preceding observations are Reissner-Nordström spacetimes (section 4.1). We calculate the static and infalling quasilocal energies and the Hamiltonian for such spacetimes. We show how these quantities correspond to the energies that we would expect using a quasi-Newtonian intuition. We also see that infalling observers will measure arbitrarily large total energies as they fall into black holes that are arbitrarily close to being extreme. This contrasts with observers measuring the geometric part of the energy, who will find results only slightly larger in magnitude than those seen by their static counterparts.

Having used the Reissner-Nordström spacetimes to orient ourselves and gain some intuition, in section 4.2 we finally apply the non-orthogonal formalism to our intended target – the naked black holes. There we find that the behaviour of quasilocal quantities calculated for these black holes is qualitatively the same as for near extreme Reissner-Nordström holes with one exception. For reasonable choices of the coupling constant between the electromagnetic and dilaton fields, static observers measuring the geometric energy will record large values (proportional to the mass of the hole) while those who are infalling will actually measure arbitrarily small energies as they cross the horizon. We discuss this rather surprising result in the final section of our paper.

First though, as a preliminary to all of the above work, we establish a set of definitions and examine the regions of spacetime that we shall be working with (section 2.1) and then review the field equations of electromagnetism coupled to a dilaton field on a general relativistic background (section 2.2).

2 Definitions and Set-up

Apart from some minor changes, most of the notation for this paper will be familiar to those who have read [6] and [24]. In this section we set up the notation and review some facts on matter fields that will be pertinent to the rest of the work. We begin by setting up the geometrical background in which we will be working.

2.1 The geometry

Let $\mathcal{M}$ be a four dimensional spacetime with metric tensor field $g_{\alpha\beta}$ and in that spacetime define a smooth timelike vector field T^α and a spacelike three dimensional hypersurface Σ_0. This field and surface are sufficient to (at least locally) define a notion of time over $\mathcal{M}$; if Σ_0 is taken to be an "instant" in time then past and future "instants" may be constructed by evolving Σ_0 with the flow defined by the vector field T^α. The resultant time foliation of $\mathcal{M}$ may be parameterized by a coordinate t such that $T^\alpha \partial_\alpha t = 1$. Then in the usual way we say that an "instant" Σ_{t1} "happens" before Σ_{t2} if $t_1 < t_2$. If we consider T^a to guide the evolution of a set of observers in $\mathcal{M}$ then this is the observer defined notion of time.

We may break up T^α into its components perpendicular and parallel to the Σ_t by defining a lapse function N and a shift vector field V^α so that

$$T^\alpha = N u^\alpha + V^\alpha, \tag{1}$$

where u^α is defined so that at each point in $\mathcal{M}$ it is the future pointing unit normal vector to the appropriate hypersurface Σ_t. The shift vector is constructed so that $V^\alpha u_\alpha = 0$. The lapse and shift then tell us how observers being swept along with the time flow T^α move through space and time relative to the foliation.

If we consider a set of observers forming a closed two surface Ω_0 in Σ_0 then we may naturally define a four dimensional region $M \in \mathcal{M}$. Let B be the timelike surface defined by evolving Ω_0 using the vector field T^α. We use the foliation of M to induce a corresponding foliation of B; that is we define $\Omega_t = \Sigma_t \cap B$. Assigning one side of Ω_0 to be an "inside" and the other an "outside", we may extend that notion over all of B. Finally, choosing Σ_{t1} and Σ_{t2} as initial and final surfaces we may naturally define the region $M \in \mathcal{M}$ to be the region of spacetime inside B and that "happens" between times t_1 and t_2. Figure 1 illustrates these concepts for a three dimensional $\mathcal{M}$.

Note that while a Σ_t uniquely specifies an Ω_t, the converse isn't true. Any number of Σ_t foliations can be defined that are compatible with a given Ω_t foliation. In fact despite the way that we have set up the foliations in this section, for most of this paper we will only be concerned with the foliation Ω_t. The foliation of the rest of the spacetime is irrelevant, basically because there are no observers in the interior of B to define it. The only observers reside on the boundary B.

We define unit normal vector fields for the various hypersurfaces. Already we have defined u^α as the future-pointing timelike unit normal vector field to the Σ_t surfaces. Similarly, we may define $\tilde{u}^\alpha$ as the future-pointing timelike unit normal vector field to the surfaces Ω_t tangent to the hypersurface B. The spacelike outward-pointing unit normal vector field to B is defined as n^α. Then, by construction $\tilde{u}^\alpha n_\alpha = 0$ and $T^\alpha n_\alpha = 0$. We further define $\tilde{n}^\alpha$ as the vector field defined on B such that $\tilde{n}^\alpha$ is the unit normal vector to Ω_t tangent to Σ_t (Ω_t being viewed as a surface in Σ_t). By construction $u^\alpha \tilde{n}_\alpha = 0$.

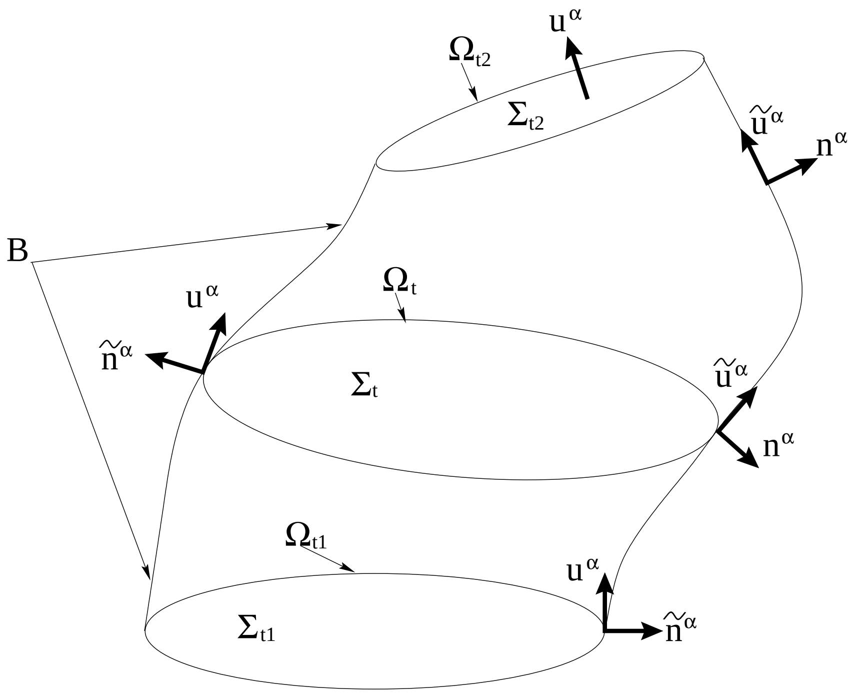

Figure 1: A three dimensional schematic of the Lorentzian region M, assorted normal vector fields, and typical elements of the foliation.

We define the scalar field $\eta = u^\alpha n_\alpha$ over B. If $\eta = 0$ everywhere, then the foliation surfaces are orthogonal to the boundary B (the case dealt with in refs. [6, 18] [1]), and the vector fields with the tildes are equal to their counterparts without tildes. If $\eta \neq 0$, we express $\tilde{u}^\alpha$ and n^α in terms of u^α and $\tilde{n}^\alpha$ (or vice versa) as,

$$n^\alpha = \frac{1}{\lambda}\tilde{n}^\alpha - \eta u^\alpha \quad \text{and} \quad \tilde{u}^\alpha = \frac{1}{\lambda}u^\alpha - \eta \tilde{n}^\alpha, \tag{2}$$

or,

$$\tilde{n}^\alpha = \frac{1}{\lambda}n^\alpha + \eta \tilde{u}^\alpha \quad \text{and} \quad u^\alpha = \frac{1}{\lambda}\tilde{u}^\alpha + \eta n^\alpha, \tag{3}$$

where $\lambda^2 \equiv \frac{1}{1+\eta^2}$.

Note too that on the surface B we may write,

$$T^\alpha = \tilde{N}\tilde{u}^\alpha + \tilde{V}^\alpha, \tag{4}$$

where we call $\tilde{N} \equiv \lambda N$ the boundary lapse and $\tilde{V}^\alpha \equiv \sigma^\alpha{}_\beta V^\beta$ the boundary shift. This is possible because we have assumed that $T^\alpha n_\alpha = 0$ on B.

Next consider the metrics induced on the hypersurfaces by the spacetime metric $g_{\alpha\beta}$. These may be written in terms of $g_{\alpha\beta}$ and the normal vector fields. $h_{\alpha\beta} \equiv g_{\alpha\beta} + u_\alpha u_\beta$ is the metric induced on the Σ_t surfaces, $\gamma_{\alpha\beta} \equiv g_{\alpha\beta} - n_\alpha n_\beta$ is the metric induced on B, and $\sigma_{\alpha\beta} \equiv h_{\alpha\beta} - \tilde{n}_\alpha \tilde{n}_\beta = \gamma_{\alpha\beta} + \tilde{u}_\alpha \tilde{u}_\beta$ is the metric induced on Ω_t. By raising one index of these metrics we obtain projection operators into the corresponding surfaces. These have the expected properties: $h^\alpha{}_\beta u^\beta = \gamma^\alpha{}_\beta n^\beta = \sigma^\alpha{}_\beta n^\beta = \sigma^\alpha{}_\beta u^\beta = 0$, and $h^\alpha{}_\beta h^\beta{}_\gamma = h^\alpha{}_\gamma$, $\gamma^\alpha{}_\beta \gamma^\beta{}_\gamma = \gamma^\alpha{}_\gamma$, and $\sigma^\alpha{}_\beta \sigma^\beta{}_\gamma = \sigma^\alpha{}_\gamma$.

On choosing a coordinate system $\{x^1, x^2, x^3\}$ on the surface Σ_0 we define $h = \det(h_{\alpha\beta})$ (where in this case we take $h_{\alpha\beta}$ as the coordinate representation of that metric tensor). We then map this coordinate system to each of the other Σ_t surfaces by Lie-dragging with the vector field u^α;

[1] The definitions of u^α and n^α are consistent with ref. [6], but interchanged with respect to those in ref. [18].

combining this set of coordinates on each surface with the time coordinate $x^0 \equiv t$ we have a coordinate system over all of M. We define $g = \det(g_{\alpha\beta})$. Similarly, choosing a coordinate system on Ω_t we define $\sigma = \det(\sigma_{\alpha\beta})$. Again, using the time flow to extend the coordinate system over all of B, we define $\gamma = \det(\gamma_{\alpha\beta})$. It is then not hard to show [23] that

$$\sqrt{-g} = N\sqrt{h} \text{ and } \sqrt{-\gamma} = \tilde{N}\sqrt{\sigma}. \tag{5}$$

We also define the following extrinsic curvatures. Taking ∇_α as the covariant derivative on $\mathcal{M}$ compatible with $g_{\alpha\beta}$, the extrinsic curvature of Σ_t in $\mathcal{M}$ is $K_{\alpha\beta} \equiv -h^\gamma{}_\alpha h^\delta{}_\beta \nabla_\gamma u_\delta = -\frac{1}{2}\pounds_u h_{\alpha\beta}$, where $\pounds_u$ is the Lie derivative in the direction u^α. The extrinsic curvature of B in $\mathcal{M}$ is $\Theta_{\alpha\beta} = -\gamma^\gamma{}_\alpha \gamma^\delta{}_\beta \nabla_\gamma n_\delta$ while the extrinsic curvature of Ω_t in Σ_t is $k_{\alpha\beta} \equiv -\sigma^\gamma{}_\alpha \sigma^\delta{}_\beta \nabla_\gamma \tilde{n}_\delta$. Contracting each of these with the appropriate metric we define $K \equiv h^{\alpha\beta}K_{\alpha\beta}$, $\Theta \equiv \gamma^{\alpha\beta}\Theta_{\alpha\beta}$, and $k \equiv \sigma^{\alpha\beta}k_{\alpha\beta}$.

Finally, we define the following intrinsic quantities over $\mathcal{M}$ and Σ_t. On $\mathcal{M}$, the Ricci tensor, Ricci scalar, and Einstein tensor are $\mathcal{R}_{\alpha\beta}$, $\mathcal{R}$, and $G_{\alpha\beta}$ respectively. $\epsilon_{\alpha\beta\gamma\delta}$ is the completely skew symmetric Levi-Cevita tensor. On Σ_t, D_α is the covariant derivative compatible with $h_{\alpha\beta}$, while $R_{\alpha\beta}$ and R are respectively the intrinsic Ricci tensor and scalar.

2.2 Fields on the spacetime

We consider spacetimes containing a cosmological constant Λ, a massless scalar field ϕ (the dilaton), and a Maxwell field $F_{\alpha\beta}$. The field equations are:

$$\begin{aligned} \frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\nabla_\beta F_{\gamma\delta} &= 0, & (6)\\ \nabla_\beta(e^{-2a\phi}F^{\alpha\beta}) &= 0, & (7)\\ \nabla^\alpha\nabla_\alpha\phi + \frac{1}{2}ae^{-2a\phi}F_{\alpha\beta}F^{\alpha\beta} &= 0, \text{ and} & (8)\\ G_{\alpha\beta} + \Lambda g_{\alpha\beta} - 8\pi T_{\alpha\beta} &= 0, & (9) \end{aligned}$$

where a is the coupling constant between the scalar and Maxwell fields, and

$$T_{\alpha\beta} \equiv \frac{1}{4\pi}\left([\nabla_\alpha\phi][\nabla_\beta\phi] - \frac{1}{2}[\nabla^\gamma\phi][\nabla_\gamma\phi]g_{\alpha\beta} + e^{-2a\phi}[F_{\alpha\gamma}F_\beta{}^\gamma - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta}]\right) \tag{10}$$

is the stress-energy tensor associated with the matter. The first equation holds because the electromagnetism is described by a gauge field, while the last three may be derived from the action principle as we will do in section 3. However we first review some useful facts regarding these fields and their projection into three dimensional hypersurfaces.

First define the dual $\star F_{\alpha\beta} = \frac{1}{2}e^{-2a\phi}\epsilon_{\alpha\beta}{}^{\gamma\delta}F_{\gamma\delta}$ of $F_{\alpha\beta}$. Then, we may respectively rewrite the above equations as

$$\begin{aligned} \nabla_\beta(e^{2a\phi}\star F^{\alpha\beta}) &= 0, & (11)\\ -\frac{1}{2}\epsilon^{\alpha\beta\gamma\delta}\nabla_\beta\star F_{\gamma\delta} &= 0, & (12)\\ \nabla^\alpha\nabla_\alpha\phi - \frac{1}{2}ae^{2a\phi}\star F_{\alpha\beta}\star F^{\alpha\beta} &= 0, \text{ and} & (13)\\ G_{\alpha\beta} + \Lambda g_{\alpha\beta} - 8\pi T_{\alpha\beta} &= 0, & (14) \end{aligned}$$

where this time we express the stress energy as

$$T_{\alpha\beta} = \frac{1}{4\pi}\left([\nabla_\alpha\phi][\nabla_\beta\phi] - \frac{1}{2}[\nabla^\gamma\phi][\nabla_\gamma\phi]g_{\alpha\beta} + e^{2a\phi}[\star F_{\alpha\gamma}\star F_\beta{}^\gamma - \frac{1}{4}g_{\alpha\beta}\star F_{\gamma\delta}\star F^{\gamma\delta}]\right). \tag{15}$$

Thus, the equations of motion as a set are invariant under the duality transformation ($\phi \to -\phi, F_{\alpha\beta} \to \star F_{\alpha\beta}$).

Second we define projections of the Maxwell $U(1)$ field into the spacelike hypersurfaces Σ_t considered above. If u^α is the normal to that surface, then the (dilaton modified) electric and magnetic fields in that surface are $E_\alpha \equiv e^{-2a\phi} F_{\alpha\beta} u^\beta$ and $B_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta} u^\beta F_{\gamma\delta}$. Then, a simple calculation shows that $E_\alpha = -\frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta} u^\beta \star F_{\gamma\delta}$ and $B_\alpha = -e^{2a\phi} \star F_{\alpha\beta} u^\beta$. Conversely we may write $F_{\alpha\beta}$ and $\star F_{\alpha\beta}$ in terms of the electric and magnetic field three-vectors as

$$F_{\alpha\beta} = e^{2a\phi}(u_\alpha E_\beta - u_\beta E_\alpha) - \epsilon_{\alpha\beta}{}^{\gamma\delta} B_\gamma u_\delta \text{ and} \tag{16}$$

$$\star F_{\alpha\beta} = -e^{-2a\phi}(u_\alpha B_\beta - u_\beta B_\alpha) - \epsilon_{\alpha\beta}{}^{\gamma\delta} E_\gamma u_\delta \tag{17}$$

Combining these two versions of $F_{\alpha\beta}$ and $\star F_{\alpha\beta}$ with equations (7) and (11) and decomposing into the components parallel and perpendicular to the Σ_t surfaces one recovers the (dilaton modified) three dimensional source-free Maxwell equations. In particular the components parallel to u^α give us the divergence relations

$$D_\alpha B^\alpha = 0 \text{ and } D_\alpha E^\alpha = 0. \tag{18}$$

Note that in terms of the electric and magnetic field, the duality transform $F_{\alpha\beta} \to \star F_{\alpha\beta}$ becomes $E_\alpha \to -B_\alpha$ and $B_\alpha \to E_\alpha$.

Finally $F_{\alpha\beta}$ is a 2-form field and by equation (6) its exterior derivative is zero. Therefore (at least locally) there exists a one form field A_α such that $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$. This field is the electromagnetic potential. We decompose it into its components parallel and perpendicular to Σ_t as a scalar potential $\Phi \equiv -A_\alpha u^\alpha$, and three-vector potential $\bar{A}_\alpha \equiv h_\alpha^\beta A_\beta$, where Φ is the Coulomb potential and $\bar{A}_\alpha$ is the vector potential. Then the electric and magnetic fields may be written as

$$E_\alpha = -e^{-2a\phi}\left(\frac{1}{N} D_\alpha(N\Phi) + \pounds_u \bar{A}_\alpha\right) \text{ and} \tag{19}$$

$$B_\alpha = -\epsilon_{\alpha\beta}{}^{\gamma\delta} u^\beta D_\gamma \bar{A}_\delta. \tag{20}$$

Similarly by equation (6) there locally exists a 1-form $\star A_\alpha$ such that $\star F_{\alpha\beta} = \partial_\alpha \star A_\beta - \partial_\beta \star A_\alpha$. We can decompose it into components $\star\Phi = -\star A_\alpha u^\alpha$ and $\star\bar{A}_\alpha = h_\alpha^\beta \star A_\beta$ [2]. Then in terms of these potentials the electric and magnetic fields may be written as

$$E_\alpha = -\epsilon_{\alpha\beta}{}^{\gamma\delta} u^\beta D_\gamma \star\bar{A}_\delta \text{ and} \tag{21}$$

$$B_\alpha = e^{2a\phi}\left(\frac{1}{N} D_\alpha(N \star\Phi) + \pounds_u \star\bar{A}_\alpha\right). \tag{22}$$

3 Quasilocal quantities from the action principle

In this section we start from an action principle and define quasilocal energy, momentum, and stress tensor densities in the spirit of [6, 24]. We begin by reviewing the matter free case discussed in [24] and then include coupled dilaton and electromagnetic fields. We then see that the formalism that we have set up does not allow for magnetic charges and so use electromagnetic duality to modify it so that we can study magnetically charged spacetimes. Finally in the last part we see how the quasilocal quantities transform with the motion of the observers.

[2] There is an abuse of notation here. $\star A_\alpha$, $\star\bar{A}_\alpha$, and $\star\Phi$ are in no sense duals of their unstarred counterparts. The $\star$ simply indicates their relationship to $\star F_{\alpha\beta}$.

3.1 The matter free case

Given $M \subset \mathcal{M}$ with the non-orthogonal boundaries described above an appropriate action for pure gravity (with a cosmological constant) is [21]:

$$
\begin{aligned}
I &= \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}(\mathcal{R}-2\Lambda) + \frac{1}{\kappa}\int_\Sigma d^3x\sqrt{h}K - \frac{1}{\kappa}\int_B d^3x\sqrt{-\gamma}\Theta \\
&\quad + \frac{1}{\kappa}\int_\Omega d^2x\sqrt{\sigma}\sinh^{-1}(\eta) - \underline{I},
\end{aligned} \tag{23}
$$

where $\int_\Sigma = \int_{\Sigma_2 - \Sigma_1}$ and $\int_\Omega = \int_{\Omega_2} - \int_{\Omega_1}$, and if we choose a system of units where $c = G = 1$, $\kappa = 8\pi$. In general $\underline{I}$ is a functional of the boundary metrics on ∂M. It will be discussed in more detail in section 3.4; for now we take it to be zero.

As was shown in [24] the variation of I with respect to the metric is,

$$
\begin{aligned}
\delta I &= \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}(G_{\alpha\beta}+\Lambda g_{\alpha\beta})\delta g^{\alpha\beta} \\
&\quad + \int_\Sigma d^3x\left(P_h^{\alpha\beta}\delta h_{\alpha\beta}\right) + \int_\Omega d^2x\left(P_{\sqrt{\sigma}}\delta\sqrt{\sigma}\right) \\
&\quad - \int dt\int_{\Omega_t} d^2x\sqrt{\sigma}\left[\tilde{\varepsilon}\delta\tilde{N} - \tilde{\jmath}_\alpha\delta\tilde{V}^\alpha - \frac{\tilde{N}}{2}\tilde{s}^{\alpha\beta}\delta\sigma_{\alpha\beta}\right].
\end{aligned} \tag{24}
$$

In the above, $P_h^{\alpha\beta} \equiv \frac{\sqrt{h}}{2\kappa}\left(Kh^{\alpha\beta} - K^{\alpha\beta}\right)$, $P_{\sqrt{\sigma}} \equiv \frac{1}{\kappa}\sinh^{-1}\eta$, $\tilde{\varepsilon} \equiv \frac{1}{\kappa}\tilde{k}$, $\tilde{\jmath}_\alpha \equiv \frac{1}{\kappa}\sigma_\alpha{}^\beta\tilde{u}^\delta\nabla_\beta n_\delta$, and $\tilde{s}^{\alpha\beta} \equiv \frac{1}{\kappa}(\tilde{k}^{\alpha\beta} - [\tilde{k} - n^\gamma\tilde{a}_\gamma]\sigma^{\alpha\beta})$. $\tilde{a}^\alpha \equiv \tilde{u}^\beta\nabla_\beta\tilde{u}^\alpha$ is the acceleration vector associated with $\tilde{u}^\alpha$.

In the usual way if we solve $\delta I = 0$ while holding the boundary metrics constant, we recover the Einstein equations. Examining the initial and final hypersurfaces Σ_1 and Σ_2 and their boundaries Ω_1 and Ω_2, $P_h^{\alpha\beta}$ is the Σ_t hypersurface momentum conjugate to $h_{\alpha\beta}$ while $P_{\sqrt{\sigma}}$ is the Ω_t hypersurface momentum conjugate to $\sqrt{\sigma}$. Further, $-\sqrt{\sigma}\tilde{\varepsilon}$ is conjugate to the boundary lapse $\tilde{N}$, $\sqrt{\sigma}\tilde{\jmath}_\alpha$ is conjugate to the boundary shift $\tilde{V}^\alpha$, and $\frac{\tilde{N}}{2}\sqrt{\sigma}\tilde{s}^{\alpha\beta}$ is conjugate to the boundary metric $\sigma_{\alpha\beta}$. Following the Hamilton-Jacobi analysis of [6] we identify these three quantities as the surface energy, momentum, and stress densities. Note that they depend only on the foliation of the boundary and are indifferent to the foliation of the spacetime as a whole.

I may also be decomposed with respect to the foliation to obtain [24]

$$
\begin{aligned}
I &= \int_M d^4x\left(P^{\alpha\beta}\pounds_T h_{\alpha\beta} - N\mathcal{H} - V^\alpha H_\alpha\right) + \int dt\int_{\Omega_t} d^2x P_{\sqrt{\sigma}}(\pounds_T\sqrt{\sigma}) \\
&\quad - \int dt\int_{\Omega_t} d^2x\sqrt{\sigma}\left(\tilde{N}\tilde{\varepsilon} - \tilde{V}^\alpha\tilde{\jmath}_\alpha\right).
\end{aligned} \tag{25}
$$

$\mathcal{H}$ and $\mathcal{H}_\alpha$ are the Hamiltonian and momentum constraints for the gravitational field on the Σ_t surfaces and are defined as usual by

$$
\mathcal{H} \equiv -\frac{\sqrt{h}}{\kappa}(G_{\alpha\beta}+\Lambda g_{\alpha\beta})u^\alpha u^\beta = -\frac{\sqrt{h}}{2\kappa}\left(R - 2\Lambda + K^2 - K_{\alpha\beta}K^{\alpha\beta}\right) = 0, \tag{26}
$$

and

$$
\mathcal{H}_\alpha \equiv \frac{\sqrt{h}}{\kappa}h_\alpha{}^\beta(G_{\beta\gamma}+\Lambda g_{\beta\gamma})u^\gamma = \frac{\sqrt{h}}{\kappa}\left(D_\beta K_\alpha{}^\beta - D_\alpha K\right) = 0. \tag{27}
$$

They are zero for solutions to the Einstein equations.

The quasilocal quantities are exactly those that would be obtained by the analysis of [6] if the foliation Ω_t of B had been induced by a foliation Σ_t of M that was perpendicular to B. Consequently we may think of them as being defined for a (local) orthogonal extension of the foliation of B into M. These are the natural quantities that a set of observers restricted to the surface B would measure. As was mentioned earlier, such observers know about the foliation of the boundary (it is defined by their notion of simultaneity) but being restricted to B they have no unique way of associating that foliation with a foliation of M as a whole. The natural foliation for them to work with is therefore the (local) orthogonal extension of B. In turn, this means that they will measure the quantities that we have seen arise from the action in a natural way.

An observer-dependent Hamiltonian may also be defined. Recall that in elementary classical mechanics with one degree of freedom, the action I and Hamiltonian H are related by the equation $I = \int dt(p\dot{q} - H)$, where q is the configuration variable for the system and p is the momentum. Extending this definition of the Hamiltonian to the system under consideration we obtain the Hamiltonian

$$H = \int_{\Sigma_t} d^3x[N\mathcal{H} + V^\alpha H_\alpha] + \int_{\Omega_t} d^2x\sqrt{\sigma}(\tilde{N}\tilde{\varepsilon} - \tilde{V}^\alpha \tilde{\jmath}_\alpha), \tag{28}$$

for a spatial three surface Σ_t bounded by Ω_t. If T^α is a Killing vector of the boundary metric $\gamma_{\alpha\beta}$, then this Hamiltonian is a fixed charge. It is usually taken to be the mass contained within Ω_t [6, 8]. Note that for solutions to the equations of motion the bulk constraint terms are zero and so the Hamiltonian depends only on the foliation of the boundary.

Finally, before we move on to include matter fields, note that the above derivation assumes that within M the metric tensor is non-singular. If this is not the case, then the frequent uses of Stokes theorem in the derivation don't apply. Of course, black holes do have singularities in their centers and so problems arise if we wish to study them. For black holes, the full analysis only applies if B includes an inner boundary B' cutting off the singularity – in which case we must also consider the boundary terms on B' and are in fact considering the energy contained between the two bounding surfaces rather than in the hole itself. On the other hand if we consider a star there are no such singularities and it is reasonable to consider just the outer boundary B. Thus, in order to make a comparison between stars and black holes (which after all are described by the same solutions to the Einstein equations once you are beyond their defining surfaces) it is conventional to go beyond the preceding derivation and define quasilocal energies (and momenta) for black holes using just an outer boundary B. This is equivalent to the way one defines the electric charge of a point charge (as opposed to a charge distribution) in electromagnetism.

There is an alternative but entirely equivalent way to look at this for spacetimes with an asymptotic region. Let us assume that the quasilocal energy E_∞ in the asymptotic limit is the total energy in the spacetime. As pointed out in the introduction, in this limit it does agree with most of the popular ways of defining total energy and so this is a reasonable assumption. Then for a given surface Ω_t in a leaf Σ_t we may calculate the QLE contained between Ω_t and the asymptotic surface at infinity. It is $E_\infty - E_\Omega$. This region has no singularities to cause trouble and so the analysis may be executed rigorously. Next, the quasilocal quantities are additive so the energy contained within Ω_t is $E_\infty - (E_\infty - E_\Omega) = E_\Omega$ which is the regular QLE. As noted the assumptions behind this approach are completely equivalent to those behind the one we discussed above, but it does give us a slightly different way of looking at things and highlighting what those assumptions really are.

3.2 Including coupled electromagnetic and dilaton fields

The above analysis may be extended in a straightforward matter to include gauge and dilaton fields. This extension (for the orthogonal case) was made quite generally in ref. [9] but for our purposes we just need to consider the coupled Maxwell and dilaton fields governed by the following action:

$$I_{EMdil} = \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}(\mathcal{R} - 2\Lambda - 2(\nabla_\alpha\phi)(\nabla^\alpha\phi) - e^{-2a\phi}F_{\alpha\beta}F^{\alpha\beta}) \tag{29}$$

$$+\frac{1}{\kappa}\int_\Sigma d^3x\sqrt{h}K - \frac{1}{\kappa}\int_B d^3x\sqrt{-\gamma}\Theta + \frac{1}{\kappa}\int_\Omega d^2x\sqrt{\sigma}\sinh^{-1}(\eta) - \underline{I}.$$

For the rest of this section we shall assume that there exists a single (but not unique) vector potential A_α properly defined over all of M such that $F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha$ (the converse of this assumption is that multiple vector potentials are required to generate F, each of which is defined over only part of M). As we shall see in section 3.3 this is equivalent to assuming that there is no magnetic charge contained by any closed surface in M.

Then, it is a simple calculation to show that

$$\begin{aligned} & \delta\left(-2\sqrt{-g}(\nabla_\alpha\phi)(\nabla^\alpha\phi) - \sqrt{-g}e^{-2a\phi}F_{\alpha\beta}F^{\alpha\beta}\right) \qquad (30)\\ = \; & 4\sqrt{-g}\mathcal{F}_{Dil}\delta\phi + 4\sqrt{-g}\mathcal{F}^\beta_{EM}\delta A_\beta - \kappa\sqrt{-g}T_{\alpha\beta}\delta g^{\alpha\beta} \\ & -4\sqrt{-g}\nabla_\alpha\left([\nabla^\alpha\phi]\delta\phi + e^{-2a\phi}F^{\alpha\beta}\delta A_\beta\right). \end{aligned}$$

Further,

$$\mathcal{F}_{Dil} = \nabla^\alpha\nabla_\alpha\phi + \frac{1}{2}ae^{-2a\phi}F_{\alpha\beta}F^{\alpha\beta} \qquad (31)$$

$$\mathcal{F}^\beta_{EM} = \nabla_\alpha[e^{-2a\phi}F^{\alpha\beta}], \text{ and} \qquad (32)$$

$$T_{\alpha\beta} = \frac{2}{\kappa}\left([\nabla_\alpha\phi][\nabla_\beta\phi] - \frac{1}{2}[\nabla^\gamma\phi][\nabla_\gamma\phi]g_{\alpha\beta}\right) + e^{-2a\phi}T^{EM}_{\alpha\beta}, \qquad (33)$$

where $T^{EM}_{\alpha\beta} = \frac{2}{\kappa}(F_{\alpha\gamma}F_\beta{}^\gamma - \frac{1}{4}g_{\alpha\beta}F_{\gamma\delta}F^{\gamma\delta})$ is the standard electromagnetic stress energy tensor.

The final term of equation (30) is a total divergence, and as such when we substitute it into the expression for δI_{EMdil} may be moved out to the boundary using Stokes theorem (here we make use of the assumption that there is a single A_α). Then, using the vacuum result (24) we obtain

$$\begin{aligned} \delta I_{EMdil} = \; & \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\left\{(G_{\alpha\beta} + \Lambda g_{\alpha\beta} - 8\pi T_{\alpha\beta})\delta g^{\alpha\beta} + 4\mathcal{F}_{Dil}\delta\phi + 4\mathcal{F}^\beta_{EM}\delta A_\beta\right\} \qquad (34)\\ & + \int_\Sigma d^3x\left\{P^{\alpha\beta}_h\delta h_{\alpha\beta} + P_\phi\delta\phi + P^\alpha_{\bar{A}}\delta\bar{A}_\alpha\right\} + \int_\Omega d^2x\left\{P_{\sqrt{\sigma}}\delta(\sqrt{\sigma})\right\} \\ & - \int dt\int_{\Omega_t} d^2x\sqrt{\sigma}\left\{(\tilde{\varepsilon} + \tilde{\varepsilon}^m)\delta\tilde{N} - (\tilde{\jmath}_\alpha + \tilde{\jmath}^m_\alpha)\delta\tilde{V}^\alpha - \frac{\tilde{N}}{2}s^{\alpha\beta}\delta\sigma_{\alpha\beta}\right\} \\ & + \frac{2}{\kappa}\int dt\int_{\Omega_t} d^2x\sqrt{\sigma}\tilde{N}\left\{-\pounds_n\phi\delta\phi + (n_\beta\tilde{E}^\beta)\delta\tilde{\Phi} - e^{-2a\phi}\tilde{u}^\alpha n^\beta\epsilon_{\alpha\beta}{}^{\gamma\delta}\tilde{B}_\gamma\delta\hat{A}_\delta\right\}. \end{aligned}$$

In the above, $P^{\alpha\beta}_h \equiv \frac{\sqrt{h}}{2\kappa}(Kh^{\alpha\beta} - K^{\alpha\beta})$ and $P_{\sqrt{\sigma}} \equiv \frac{\sinh^{-1}(\eta)}{\kappa}$ as before while $P_\phi \equiv \frac{2\sqrt{h}}{\kappa}\pounds_u\phi$ and $P^\alpha_{\bar{A}} \equiv -\frac{2\sqrt{h}}{\kappa}E^\alpha$ (recall that $\bar{A}_\alpha = h^\beta_\alpha A_\beta$). They are the boundary momentum tensor densities conjugate to $h_{\alpha\beta}$, ϕ, $\bar{A}_\alpha$, and $\sqrt{\sigma}$ respectively. $\sqrt{\sigma}\varepsilon$, $\sqrt{\sigma}\tilde{\jmath}^\alpha$, and $\sqrt{\sigma}\frac{\tilde{N}}{2}s^{\alpha\beta}$ are again the surface energy, angular momentum, and stress densities associated with pure Einstein gravity as defined in the previous section, while $\sqrt{\sigma}\varepsilon^m$ and $\sqrt{\sigma}\tilde{\jmath}^\alpha_m$ are the extra bits of surface energy and angular momentum density that must be added on to allow for the presence of the matter. They are

$$\tilde{\varepsilon}^m \equiv \frac{2}{\kappa}n_\beta\tilde{E}^\beta\tilde{\Phi} \quad \text{and} \quad \tilde{\jmath}^m_\alpha \equiv \frac{2}{\kappa}n_\beta\tilde{E}^\beta\hat{A}_\alpha. \qquad (35)$$

$\tilde{\Phi} \equiv -A_\alpha\tilde{u}^\alpha$ is the Coulomb potential and $\tilde{E}_\alpha \equiv e^{-2a\phi}F_{\alpha\beta}\tilde{u}^\beta$ is the electric field associated with $\tilde{u}^\alpha$, the timelike normal to Ω_t in B. $\hat{A}_\alpha \equiv \sigma^\beta_\alpha A_\alpha$ is the projection of the vector potential into the Ω_t two boundaries and $\tilde{B}_\alpha \equiv -\frac{1}{2}\epsilon_{\alpha\beta}{}^{\gamma\delta}\tilde{u}^\beta F_{\gamma\delta}$.

Consider variations that fix $h_{\alpha\beta}$, ϕ, and $\bar{A}_\alpha$ on the spacelike boundaries, and fix $\gamma_{\alpha\beta}$ (equivalently $\tilde{N}$, $\tilde{V}^\alpha$, $\sigma_{\alpha\beta}$), ϕ, and $\gamma_\alpha^\beta A_\beta$ (equivalently $\tilde{\Phi}$ and $\sigma_\alpha^\beta A_\beta$) on the timelike boundaries. Then solving $\delta I_{EMdil} = 0$ we obtain the equations of motion (7,8,9) as promised in section 2.2. Equivalently this particular action is only fully differentiable if we *a priori* fix all of the boundary metrics plus ϕ and $\gamma_\alpha^\beta A_\beta$ on the timelike boundary.

It is not hard to see that fixing $\tilde{\Phi}$ and $\hat{A}_\beta$ on the timelike boundaries is equivalent to fixing the component of $\tilde{B}^\alpha$ perpendicular to B (that is $\tilde{B}^\alpha n_\alpha$) and the components of $\tilde{E}^\alpha$ parallel to Ω_t (that is $\sigma_\beta^\alpha \tilde{E}^\beta$). By Gauss's law the integral of $\tilde{B}^\alpha n_\alpha$ over Ω_t is the magnetic charge contained by that surface. Thus the action is fully differentiable only if we restrict the parameter space of possible solutions to those with a specified magnetic charge which fits in nicely with our previous comment that the magnetic charge must be zero by our assumption that there exists a single vector potential generating the EM fields. In contrast, there is no restriction on the electric charge. We will further investigate this inequivalent treatment of the electric and magnetic charges in section 3.3.

To decompose the action itself, we recalculate the gravitational constraints (this time including the matter fields) and obtain

$$\begin{aligned} \mathcal{H}^m &\equiv -\frac{\sqrt{h}}{\kappa}(G_{\alpha\beta} + \Lambda g_{\alpha\beta} - 8\pi T_{\alpha\beta})u^\alpha u^\beta \\ &= \mathcal{H} + \frac{\sqrt{h}}{\kappa}\left([D^\gamma\phi][D_\gamma\phi] + (\pounds_u\phi)^2 + e^{2a\phi}E^2 + e^{-2a\phi}B^2\right) \end{aligned} \tag{36}$$

and

$$\begin{aligned} \mathcal{H}_\alpha^m &\equiv \frac{\sqrt{h}}{\kappa}h_\alpha^\beta(G_{\beta\gamma} + \Lambda g_{\beta\gamma} - 8\pi T_{\beta\gamma})u^\gamma \\ &= \mathcal{H}_\alpha + \frac{2\sqrt{h}}{\kappa}\left([\pounds_u\phi]D_\alpha\phi + \epsilon_{\alpha\beta\gamma\delta}E^\beta B^\gamma u^\delta\right). \end{aligned} \tag{37}$$

At the same time, starting from the definition of E_α in terms of the potentials (equation 19) it is not hard to rewrite E_α as

$$E_\alpha = \frac{e^{-2a\phi}}{N}\left(D_\alpha[-N\Phi + V^\beta \bar{A}_\beta] - \pounds_T \bar{A}_\alpha + \epsilon_{\alpha\beta\gamma\delta}V^\beta B^\gamma u^\delta\right). \tag{38}$$

We may then combine these three results to decompose the action I_{EMdil} as follows. First, after decomposing the purely gravitational terms as before we are left with the following as the bulk term integrand:

$$P^{\alpha\beta}\pounds_T h_{\alpha\beta} - N\mathcal{H} - V^\alpha H_\alpha - \frac{\sqrt{-g}}{\kappa}\nabla_\alpha\phi\nabla^\alpha\phi - \frac{\sqrt{-g}}{2\kappa}e^{-2a\phi}F_{\alpha\beta}F^{\alpha\beta}. \tag{39}$$

Bringing in (36) and (37) we may then rewrite this as

$$\begin{aligned} &P^{\alpha\beta}\pounds_T h_{\alpha\beta} - N\mathcal{H}^m - V^\alpha H_\alpha^m + \frac{2\sqrt{h}}{\kappa}(N[\pounds_u\phi]^2 + V^\alpha \pounds_u\phi D_\alpha\phi) \\ &+\frac{2\sqrt{h}}{\kappa}(e^{2a\phi}NE^2 + V^\alpha\epsilon_{\alpha\beta\gamma\delta}E^\beta B^\gamma u^\delta). \end{aligned} \tag{40}$$

Next with (38) and the trivial $\pounds_T\phi = N\pounds_u\phi + V^\alpha D_\alpha\phi$ we may rewrite this entirely in terms of time derivatives of fields, constraint equations of those fields, and total divergences that may be removed to the boundaries. Then (39) becomes

$$\begin{aligned} &P^{\alpha\beta}\pounds_T h_{\alpha\beta} + P_\phi\pounds_T\phi + P_{\bar{A}}^\alpha\pounds_T\bar{A} - NH^m - V^\alpha H_\alpha^m - T^\alpha A_\alpha\mathcal{Q} \\ &+\frac{2\sqrt{h}}{\kappa}D_\beta(E^\beta T^\alpha A_\alpha). \end{aligned} \tag{41}$$

$\mathcal{Q} \equiv \frac{2\sqrt{h}}{\kappa} D_\alpha E^\alpha$ is the constraint equation for the electric field with no sources (equation 18). Thus the final result is

$$
\begin{aligned}
I_{EMdil} \;=\; & \int dt \int_{\Sigma_t} d^3x \left\{ P_h^{\alpha\beta} \pounds_T h_{\alpha\beta} + P_\phi \pounds_T \phi + P_{\bar{A}}^\alpha \pounds_T \bar{A}_\alpha \right\} \\
& + \int dt \int_{\Omega_t} d^2x \left\{ P_{\sqrt{\sigma}} (\pounds_T \sqrt{\sigma}) \right\} \\
& - \int dt \int_{\Sigma_t} d^3x \left\{ N\mathcal{H}^m + V^\alpha H_\alpha^m + T^\alpha A_\alpha \mathcal{Q} \right\} \\
& - \int dt \int_{\Omega_t} d^2x \sqrt{\sigma} \left\{ \tilde{N}(\tilde{\varepsilon} + \tilde{\varepsilon}^m) - \tilde{V}^\alpha (\tilde{\jmath}_\alpha + \tilde{\jmath}_\alpha^m) \right\}.
\end{aligned} \tag{42}
$$

Note that once again we have used Stokes theorem and therefore the assumption that there exists a single A_α defined over all of M.

Just as in section (3.1) we may define a Hamiltonian. Including the matter fields it is

$$
H^m = \int_{\Sigma_t} d^3x [N\mathcal{H}^m + V^\alpha H_\alpha^m + T^\alpha A_\alpha \mathcal{Q}] + \int_{\Omega_t} d^2x \sqrt{\sigma} \left\{ \tilde{N}(\tilde{\varepsilon} + \tilde{\varepsilon}^m) - \tilde{V}^\alpha (\tilde{\jmath}_\alpha + \tilde{\jmath}_\alpha^m) \right\}, \tag{43}
$$

for a spatial three surface Σ_t bounded by Ω_t. Again however, the Hamiltonian is actually independent of that foliation Σ_t for solutions to the equations of motion.

Note that even though the action I_{EMdil} is gauge invariant, this Hamiltonian does not necessarily inherit that invariance. The paths by which this gauge dependence can creep in are quite easily found but the effect is quite important so we will pause here to point them out in some detail. First, it is a simple matter to rewrite the action in terms of kinetic and Hamiltonian (potential) terms. Specifically,

$$
\begin{aligned}
I_{EMdil} = \int dt \; & \left\{ \int_{\Sigma_t} d^3x (P_h^{\alpha\beta} \pounds_T h_{\alpha\beta} + P_\phi \pounds_T \phi + P_{\bar{A}}^\alpha \pounds_T \bar{A}_\alpha) \right. \\
& \left. + \int_{\Omega_t} d^2x (P_{\sqrt{\sigma}} \pounds_T \sqrt{\sigma}) - H^m \right\} - \underline{I}.
\end{aligned} \tag{44}
$$

Thus while the time integrated sum of the Hamiltonian and kinetic terms must be gauge invariant, that invariance isn't necessarily inherited by the time integral of Hamiltonian itself unless part of the gauge freedom is used to ensure that $\pounds_T \bar{A}_\alpha = 0$. If this is the case and H^m is independent of the leaf of the foliation then H^m will be independent of the remaining gauge freedom. That gauge and foliation are the natural ones to choose in stationary spacetimes. Nevertheless we should keep in mind that a (partial) gauge choice has been made.

In the conventional usage of this work, there is an alternate route by which gauge dependence can also work its way into the Hamiltonian. Namely, as was discussed at the end of section 3.1, if we consider spacetimes containing singularities then the above analysis only strictly applies if we include an inner boundary to cut out that singularity. For non-singular spacetimes, there is no need to include this second boundary however, and so in the interest of comparing singular to non-singular spacetimes, it is conventional to work with the outer boundary only. Without that inner boundary however, the gauge dependence returns. In section 4.1.4 we see how this shows up in a Reissner-Nordström spacetime. There we will also see that the remaining gauge dependence resulting from neglecting the inner boundary amounts to little more than a choice of where to put the zero of the electromagnetic energy.

Thus, we see that the gauge independence of the action doesn't necessarily ensure the gauge invariance of the Hamiltonian. Specializing to the case where the lapse $N = 1$ and shift $V^\alpha = 0$ on the boundary, and leaving aside the issue of how such a choice affects the relative foliations of the inner and outer boundaries, we note that the QLE will not in general be gauge independent either.

3.3 Electromagnetic duality

In the previous section we started out by making the assumption that there was a single vector potential A_α defined over all of M. This assumption immediately means that there is no magnetic charge in M (or contained by any surface residing in M). To see this let Ω_X be any closed spatial two surface in M with normals $\tilde{u}^\alpha$ and n^α. Then, the magnetic charge contained within Ω_X is $\int_{\Omega_X} d^2x\sqrt{\sigma}n^\alpha\tilde{B}_\alpha$. By equation (20), $n^\alpha\tilde{B}_\alpha = -n^\alpha\tilde{u}^\beta\epsilon_{\alpha\beta}{}^{\gamma\delta}D_\gamma\bar{A}_\delta = d^X_\gamma(-n^\alpha\tilde{u}^\beta\epsilon_{\alpha\beta}{}^{\gamma\delta}\bar{A}_\delta)$ where d^X_α is the covariant derivative in the surface Ω_X. But this is a total derivative and so integrated over a closed surface it is zero[3]. Thus there is no magnetic charge contained by any surface in M. Thinking back to the discussion following equation (34) this agrees with the observation that the magnetic charge must be fixed under variations. Indeed it must, and that fixed value is zero. There is no room for magnetic charge within the set of assumptions that we have made.

Keep in mind that this is a stronger statement than just the local statement that $F = dA \Rightarrow dF = d(dA) = 0 \Rightarrow D_\alpha B^\alpha = 0$. When working with a vector potential, the manifestation of magnetic charge in the potential is global and topological (resulting from a twist in the $U(1)$ gauge bundle) rather than local as is the case for electric charge. If we assume that there is a single potential covering M then the $U(1)$ gauge bundle is trivial by definition and so there is no magnetic/topological charge. Even more strongly, as noted earlier, no surface contained in M can contain magnetic charge. This means, for example, that if M is the region contained within two concentric spheres (multiplied by a time interval), then not only is there no charge in M but also there is no charge in the region inside the inner sphere.

In fact, projecting into spatial slices Σ_t of M, de Rhams theorem (see for example [26]) tells us that a single vector potential is defined over all of Σ_t if and only if there is no magnetic charge contained within any two surface $\Omega_X \subset \Sigma_t$. Thus if we wish to allow magnetic charge in our spacetimes we must also break M into at least two regions each of which has its own vector potential. Then, the frequent uses of Stokes theorem in the derivation will remove total divergences to the boundaries of those regions rather than just the boundary of M itself. By definition some of those region boundaries will actually be interior to M and so observers inhabiting ∂M will not be in a position to measure all of the boundary terms and therefore will not be able to fully assess what is happening in the interior of M.

We can however study magnetically charged spacetimes if we use a duality rotation to make the magnetic charge local and analytic (while the electric charge becomes global and topological), in which case the action I_{EMdil} becomes

$$\begin{aligned} \star I_{EMdil} &= \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}(\mathcal{R} - 2\Lambda - 2(\nabla_\alpha\phi)(\nabla^\alpha\phi) - e^{2a\phi}\star F_{\alpha\beta}\star F^{\alpha\beta}) \\ &\quad + \frac{1}{\kappa}\int_\Sigma d^3x\sqrt{h}K - \frac{1}{\kappa}\int_B d^3x\sqrt{-\gamma}\Theta + \frac{1}{\kappa}\int_\Omega d^2x\sqrt{\sigma}\sinh^{-1}(\eta) - \underline{I}. \end{aligned} \tag{45}$$

Note that since $F_{\alpha\beta}F^{\alpha\beta} = -\star F_{\alpha\beta}\star F^{\alpha\beta}$ this action is not equal to I_{EMdil}. Nevertheless this is the correct action to use if we wish to obtain the usual equations of motion. To wit, assume there exists a single vector potential $\star A_\alpha$ that generates $\star F_{\alpha\beta}$ over all of M. Then, the variation of the action is

$$\begin{aligned} &\delta(\star I_{EMdil}) \\ = &\ \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}\left\{(G_{\alpha\beta} + \Lambda g_{\alpha\beta} - 8\pi T_{\alpha\beta})\delta g^{\alpha\beta} + 4\star\mathcal{F}_{Dil}\delta\phi + 4\star\mathcal{F}^\beta_{EM}\delta(\star A_\beta)\right\} \\ &+ \int_\Sigma d^3x\left\{P^{\alpha\beta}_h\delta h_{\alpha\beta} + P_\phi\delta\phi + P^\alpha_{\star\bar{A}}\delta(\star\bar{A}_\alpha)\right\} + \int_\Omega d^2x\left\{P_{\sqrt{\sigma}}\delta(\sqrt{\sigma})\right\} \end{aligned} \tag{46}$$

[3] In the more efficient differential forms notation, in the spatial slice orthogonal to $\tilde{u}^\alpha$, $\bar{A}$ is a one form and $B = d\bar{A}$ is a two form. Consequently the magnetic charge contained within Ω_X is $\int_{\Omega_X} B = \int_{\Omega_X} d\bar{A} = 0$ since Ω_X is closed.

$$
-\int dt \int_{\Omega_t} d^2x\sqrt{\sigma}\left\{(\tilde{\varepsilon}+\star\tilde{\varepsilon}^m)\delta\tilde{N}-(\tilde{\jmath}_\alpha+\star\tilde{\jmath}^m_\alpha)\delta\tilde{V}^\alpha-\frac{\tilde{N}}{2}s^{\alpha\beta}\delta\sigma_{\alpha\beta}\right\}
$$
$$
+\frac{2}{\kappa}\int dt\int_{\Omega_t} d^2x\sqrt{\sigma}\tilde{N}\left\{-\pounds_n\phi\delta\phi-(n_\beta\tilde{B}^\beta)\delta(\star\tilde{\Phi})-e^{2a\phi}\tilde{u}^\alpha n^\beta\epsilon_{\alpha\beta}{}^{\gamma\delta}\tilde{E}_\gamma\delta(\star\hat{A}_\delta)\right\},
$$

where most of the quantities retain their meanings from (34) but the starred quantities have undergone the usual transformation. Ergo, $\star\mathcal{F}_{Dil}\equiv-\nabla^\alpha\nabla_\alpha\phi+\frac{1}{2}ae^{2a\phi}\star F_{\alpha\beta}\star F^{\alpha\beta}$, $\star\mathcal{F}^\beta_{EM}\equiv\nabla_\alpha(e^{2a\phi}\star F^{\alpha\beta})$, $P_{\star\phi}-\frac{2\sqrt{h}}{\kappa}\pounds_u\phi$, $P^\alpha_{\star\bar{A}}\equiv\frac{2\sqrt{h}}{\kappa}B^\alpha$, $\star\tilde{\varepsilon}^m\equiv-\frac{2}{\kappa}n_\beta\tilde{B}^\beta\star\tilde{\Phi}$, and $\star\tilde{\jmath}^m_\alpha\equiv-\frac{2}{\kappa}(n_\beta\tilde{B}^\beta)\star\hat{A}_\alpha$. If the boundary terms are all zero we obtain the usual equations of motion. The quantities that must be held constant in order to get those boundary terms equal to zero are unchanged for the pure gravitational/geometric terms, but the electromagnetic terms have reversed so now $\tilde{E}^\alpha n_\alpha$ (and consequently the electric charge) and $\sigma^\alpha_\beta\tilde{B}^\beta$ should be fixed. Of course by our assumption about the vector potential there is no electric charge and so, as for the magnetic charge in the previous example, it is naturally fixed.

The decomposition of this action is

$$
\begin{aligned}
\star I_{EMdil} \;=\; & \int dt\int_{\Sigma_t} d^3x\left\{P^{\alpha\beta}_h\pounds_T h_{\alpha\beta}+P_{\star\phi}\pounds_T\phi+P^\alpha_{\star\bar{A}}\pounds_T(\star\bar{A}_\alpha)\right\} \\
&+\int dt\int_{\Omega_t} d^2x\left\{P_{\sqrt{\sigma}}(\pounds_T\sqrt{\sigma})\right\} \\
&-\int dt\int_{\Sigma_t} d^3x\left\{N\mathcal{H}^m+V^\alpha\mathcal{H}^m_\alpha+(T^\alpha\star A_\alpha)\star\mathcal{Q}\right\} \\
&-\int dt\int_{\Omega_t} d^2x\sqrt{\sigma}\left\{\tilde{N}(\tilde{\varepsilon}+\star\tilde{\varepsilon}^m)-\tilde{V}^\alpha(\tilde{\jmath}_\alpha+\star\tilde{\jmath}^m_\alpha)\right\}.
\end{aligned} \tag{47}
$$

$\mathcal{H}^m$ and $\mathcal{H}^m_\alpha$ are invariant under the duality transformation, but $\star\mathcal{Q}\equiv-\frac{2\sqrt{h}}{\kappa}D_\alpha B^\alpha$.

Thus we have well defined formalisms with which we can study spacetimes containing either only electric or only magnetic charges. We do not, however, have a formalism that easily handles dyonic spacetimes. Admittedly we could make a duality rotation of the action to study a spacetime with a particular dyonic charge but that would still not allow us to consider spacetimes containing multiple dyons with varying ratios of electric and magnetic charges. Furthermore, there is something fundamentally unsatisfying about having the form of the action depend on the charges contained in the spacetime. As it stands we don't have a solution for this problem and so will not consider dyonic spacetimes in this work.

Finally, we recall that the relationship between electric and magnetic charge and an $F_{\alpha\beta}F^{\alpha\beta}$ type action has been considered before in studies of the production of charged black hole pairs [27, 28, 11, 12]. In these cases one computes the probability for an empty spacetime with a source of excess energy, such as a cosmological constant, to quantum tunnel into a spacetime containing a pair of black holes. This process is calculated within the path integral formalism where the actions of instanton solutions to the equations of motion are taken as the lowest order approximation of the full Euclidean path integral. Those instantons (and indeed the set of all paths integrated over in evaluating the path integral) are required to satisfy certain boundary conditions. First, they are required to interpolate between the initial and final spacetimes matching onto these along spatial hypersurfaces, and second, boundary conditions are enforced to keep appropriate thermodynamic (quasilocal) quantities constant. The action that one uses to calculate the path integral must be chosen so that its variation fixes those same quantities. For the creation of a pair of magnetically charged nonrotating holes the magnetic charge must be fixed while in the creation of a pair of electrically charged nonrotating charged holes the electric charge must be fixed.

Thus, in those papers I_{EMdil} is used to study the creation of magnetic black holes, while

$I_{EMdil} + \Delta I_{EMdil}$ is used to study electric black holes where

$$\Delta I_{EMdil} = -\frac{2}{\kappa}\int_{\Sigma} d^3x\sqrt{h}e^{-2a\phi}u_\alpha F^{\alpha\beta}A_\beta + \frac{2}{\kappa}\int_{B} d^3x\sqrt{-\gamma}e^{-2a\phi}n_\alpha F^{\alpha\beta}A_\beta. \tag{48}$$

In line with this paper we have included a non-zero dilaton. Then the variation of $I_{EMdil}+\Delta I_{EMdil}$ fixes the electric rather than the magnetic charge – the variation of ΔI_{EMdil} switches the boundary terms to those of $I_{\star EMdil}$. This isn't really all that surprising of course since for solutions to the (in this case dilaton-modified) Maxwell equations,

$$\begin{aligned}\Delta I_{EMdil} &= \frac{1}{\kappa}\int_M d^4x\sqrt{-g}e^{-2a\phi}F_{\alpha\beta}F^{\alpha\beta} \\ &= \frac{1}{2\kappa}\int_M d^4x\sqrt{-g}(e^{-2a\phi}F_{\alpha\beta}F^{\alpha\beta} - e^{2a\phi}\star F_{\alpha\beta}\star F^{\alpha\beta}),\end{aligned} \tag{49}$$

and so for solutions $I_{EMdil} + \Delta I_{EMdil} = I_{\star EMdil}$.

In light of these observations we can clarify what is happening for various choices of the action in pair creation processes. Our approach is commensurate with that taken in the electric case: the correct action to start with is I_{EMdil}. However, as noted above, variation of I_{EMdil} implies that the electric charge is not fixed, and so a boundary term ΔI_{EMdil} must be added to correct this problem. This term is well defined since for a purely electric black hole there is a single A_α covering all of M. In a similar way a boundary term would have to be added to fix the angular momentum of the hole if we were allowing for rotation.

By the same reasoning, the correct action to start with for a magnetic black hole is $I_{\star EMdil}$. To fix the magnetic charge one would then have to add on a boundary term $\Delta I_{\star EMdil}$, defined analogously to ΔI_{EMdil}. This is equivalent to using I_{EMdil} in considering pair production because for solutions to the equations of motion $I_{\star EMdil} + \Delta I_{\star EMdil} = I_{EMdil}$. However we note that for I_{EMdil} the derivation of the QLE from the action only properly goes through if there is a single A_α defined over all of M, in which case there can be no magnetic charge.

3.4 Transformation properties and reference terms

In this section we consider how the quasilocal quantities defined above transform with respect to boosts of the observers. We will also see how the choice of a reference term $\underline{I}$ changes these transformation properties. We will only explicitly develop the transformation laws for the $F_{\alpha\beta}$ formulation. The extension to the dual formulation is trivial.

Consider a set of observers being evolved by the vector field T^α and with an instantaneous configuration Ω_t. To avoid unproductively untidy notation we will take B to be orthogonal to the foliation Σ_t. This does not reduce the generality of what follows since we have already seen that the tilded quantities defined with respect to the foliation of B are equivalent to untilded quantities defined with respect to a local foliation orthogonal to B. As before the foliation is evolved by a vector field $T^\alpha = Nu^\alpha + V^\alpha$. We next consider a second set of observers boosted with respect to the first set. They have the same instantaneous configuration Ω_t but instead are being evolved by a vector field $T^{*\alpha}$ which may be written in terms of a lapse N^*, shift $V^{*\alpha}$, and timelike unit normal $u^{*\alpha}$ to Ω_t. The spacelike normal for this set of observers is then $n^{*\alpha}$ - defined by the requirement that $n^{*\alpha}u^*_\alpha = 0$ and $\sigma_\alpha^\beta n^*_\beta = 0$. Recycling earlier notation, we define $\eta \equiv u^{*\alpha}n_\alpha$ and $\lambda \equiv \frac{1}{\sqrt{1+\eta^2}}$. Then

$$u^{*\alpha} = \frac{1}{\lambda}u^\alpha + \eta n^\alpha \text{ and } n^{*\alpha} = \frac{1}{\lambda}n^\alpha + \eta u^\alpha. \tag{50}$$

Next, consider a set of observers who are static with respect to the foliation and who measure time as their proper time (i.e. they are being evolved by the vector field $T^\alpha = u^\alpha$). Define an orthonormal frame for each of these observers $\{u^\alpha, n^\alpha, e_1^\alpha, e_2^\alpha\}$ where $e_1^\alpha, e_2^\alpha \in T\Omega_t$. Then this set of observers sees the $T^{*\alpha}$ observers as having velocity components $v_{\vdash} \equiv -\frac{T^{*\alpha}n_\alpha}{T^{*\beta}u_\beta} = \eta\lambda$, $v_1 \equiv -\frac{T^{*\alpha}e_{1\alpha}}{T^{*\beta}u_\beta} = \frac{\lambda}{N^*}V^{*\alpha}e_{1\alpha}$, and $v_2 \equiv -\frac{T^{*\alpha}e_{2\alpha}}{T^{*\beta}u_\beta} = \frac{\lambda}{N^*}V^{*\alpha}e_{1\alpha}$ in the directions n^α, e_1^α, and e_2^α respectively.

It will also be convenient to define the extrinsic curvature of Ω_t in B as $k^{\updownarrow}_{\alpha\beta} \equiv -\sigma^\gamma_\alpha \sigma^\delta_\beta \nabla_\gamma u_\delta$, which is contracted to $k^{\updownarrow} \equiv \sigma^{\alpha\beta}k^{\updownarrow}_{\alpha\beta}$. The rate of change of n^α in the direction it points is $a^{\updownarrow} \equiv n^\beta \nabla_\beta n_\alpha$. The projection of the electromagnetic vector potential is $\Phi^{\updownarrow} \equiv -A_\alpha n^\alpha$. The choice of the $\updownarrow$ superscript is meant suggest an interchange of u^α and n^α in these quantities (as compared to the same expression without the superscript). We then define versions of the quasilocal densities with u^α and n^α interchanged:

$$\varepsilon^{\updownarrow} \equiv \frac{1}{\kappa}k^{\updownarrow}, \tag{51}$$

$$\varepsilon^{m\updownarrow} \equiv \frac{2}{\kappa}(n^\beta E_\beta)\Phi^{\updownarrow}, \tag{52}$$

$$j^{\updownarrow}_\alpha \equiv \frac{1}{\kappa}\sigma^\beta_\alpha n^\delta \nabla_\beta u_\delta, \tag{53}$$

$$j^{m\updownarrow}_\alpha \equiv \frac{2}{\kappa}(n^\beta E_\beta)\hat{A}_\alpha, \text{ and} \tag{54}$$

$$s^{\updownarrow}_{\alpha\beta} \equiv \frac{1}{\kappa}\left(k^{\updownarrow}_{\alpha\beta} - [k^{\updownarrow} - u^\gamma a^{\updownarrow}_\gamma]\sigma_{\alpha\beta}\right). \tag{55}$$

The $\updownarrow$ notation has been slightly abused in the matter terms as the u^α and n^α were not interchanged in the $n_\beta E^\beta$ terms. Note too that $j^{\updownarrow}_\alpha = -j_\alpha$ and $j^{m\updownarrow}_\alpha = j^m_\alpha$.

Some of these quantities were first used in [22] in the context of defining quantities that are invariant with respect to boosts. The simplest example of such an invariant, modified for the electromagnetic field included here, is $(\varepsilon+\varepsilon^m)^2-(\varepsilon^{\updownarrow}+\varepsilon^{m\updownarrow})^2$. This is analogous to $m^2c^2 = E^2-p^2c^2$ which is an invariant for a particle with energy E and momentum p in special relativity. This suggests that we view $\varepsilon^{\updownarrow} + \varepsilon^{m\updownarrow}$ as a momentum flux through the surface Ω_t. Support for this interpretation comes when we note that

$$\sqrt{\sigma}\varepsilon^{\updownarrow} = -\frac{\sqrt{\sigma}}{2\kappa}\sigma^{\alpha\beta}\pounds_u \sigma_{\alpha\beta} = -\frac{1}{\kappa}\pounds_u \sqrt{\sigma}. \tag{56}$$

That is, $\varepsilon^{\updownarrow}$ is zero if and only if the observers don't see the area of the surface they inhabit to be changing. If the area does change then they see a momentum flux through the surface which is something we intuitively associate with motion. Note that this means that a sphere of observers moving at constant radial speed in flat space will measure a momentum flux so the notion isn't entirely in accord with intuition. Of course without reference terms such observers will also measure a non-zero quasilocal energy so this is not entirely unexpected.

By a series of straightforward calculations we obtain expressions for the quasilocal quantities seen by the $T^{\star\alpha}$ observers in terms of the quantities measured by the T^α observers. These transformation laws are

$$\begin{aligned}\varepsilon^* + \varepsilon^{m*} &\equiv -\frac{1}{\kappa}\left(\sigma^{\alpha\beta}\nabla_\alpha n^*_\beta + 2A_\alpha u^{*\alpha}(n^{*\beta}E^*_\beta)\right) \\ &= \frac{1}{\lambda}(\varepsilon + \varepsilon^m) + \eta(\varepsilon^{\updownarrow} + \varepsilon^{m\updownarrow}),\end{aligned} \tag{57}$$

$$\begin{aligned}j^*_\alpha + j^{m*}_\alpha &\equiv \frac{1}{\kappa}\left(\sigma^\beta_\alpha u^{*\gamma}\nabla_\beta n^*_\gamma + 2(n^{*\beta}E^*_\beta)\hat{A}_\alpha\right) \\ &= (j_\alpha + j^m_\alpha) - \frac{\lambda}{\kappa}\sigma^\beta_\alpha \nabla_\beta \eta, \text{ and}\end{aligned} \tag{58}$$

$$s^*_{\alpha\beta} \equiv \frac{1}{\kappa}\left(k^*_{\alpha\beta} - [k^* - n^{*\delta}a^*_\delta]\sigma_{\alpha\beta}\right)$$

$$= \frac{1}{\lambda} s_{\alpha\beta} + \eta s^{\updownarrow}_{\alpha\beta} + \frac{\lambda}{\kappa} \sigma_{\alpha\beta} u^{*\gamma} \nabla_\gamma \eta. \tag{59}$$

In certain cases, these laws simplify into very familiar forms. Assume that $\varepsilon^{\updownarrow} = \varepsilon^{m\updownarrow} = 0$ and take $T^{*\alpha} = u^{*\alpha}$ (that is the $T^{*\alpha}$ observers move only in the direction n^α perpendicular to Ω_t and use the proper time as their measure of time). Then,

$$\begin{aligned} \varepsilon^* + \varepsilon^{m*} &= \gamma \left(\varepsilon + \varepsilon^m \right), & (60) \\ j^*_\alpha + j^{m*}_\alpha &= j^*_\alpha + j^{m*}_\alpha, \text{ and} & (61) \\ s^*_{\alpha\beta} &= \gamma s_{\alpha\beta} + \frac{1}{\kappa\gamma} \sigma_{\alpha\beta} u^{*\gamma} \nabla_\gamma (\gamma v_\vdash), & (62) \end{aligned}$$

where $\gamma = \frac{1}{\sqrt{1-v_\vdash^2}} = \sqrt{1+\eta^2} = \frac{1}{\lambda}$. So, in this case the energy density transforms as we would intuitively expect from special relativity. The angular momentum density is invariant as we would also expect since the only motion is perpendicular to its direction. The stress tensor has a somewhat more complicated transformation law that is dependent on the perpendicular component of the acceleration of the second set of observers. Breaking it up into its shear $\eta_{\alpha\beta} \equiv s_{[\alpha\beta]}$ and pressure $p \equiv \sigma^{\alpha\beta} s_{\alpha\beta}$ components we obtain a little simplification. Namely, $\eta^*_{\alpha\beta} = \frac{1}{\lambda}\eta_{\alpha\beta}$ and no longer has an acceleration dependence. However, $p^\star = \frac{1}{\lambda} p + \frac{2\lambda}{\kappa} u^{*\gamma} \nabla_\gamma \eta$ and the dependence remains.

The transformation laws change if $\underline{I} \neq 0$. Physically the reasons for this are as follows. Recall [24] that the $\underline{I} \neq 0$ may be viewed as a choice of the zeros of the quasilocal quantities. The usual way to set these zeros is to choose a reference spacetime $(\underline{M}, \underline{g}_{\alpha\beta})$ in which one demands that the quasilocal quantities all measure zero. Then we define $\underline{I}$ as follows. First, one embeds $(\Omega_t, \sigma_{\alpha\beta})$ into $(\underline{M}, \underline{g}_{\alpha\beta})$ and defines a vector field $\underline{T}^\alpha$ such that: 1) $\underline{T}^\alpha \underline{T}_\alpha = T^\alpha T_\alpha$, 2) $\pounds_T \sigma_{\alpha\beta} = \pounds_{\underline{T}} \underline{\sigma}_{\alpha\beta}$ (in the sense that their projections into Ω_t are equal), and 3) the projections of $\underline{T}^\alpha$ into Ω_t are also equal (that is the boundary lapses are equal). This will not necessarily be possible for all choices of $(\Omega_t, \sigma_{\alpha\beta})$ and T^α, but will be possible in a wide range of cases. Note that satisfying these embedding conditions amounts to embedding $(\Omega_t, \sigma_{\alpha\beta})$ in $(\underline{M}, \underline{g}_{\alpha\beta})$ and requiring that T^α and $\underline{T}^\alpha$ evolve $(\Omega_t, \sigma_{\alpha\beta})$ in the same way as in M. It is also equivalent to a local (in the coordinate time sense) embedding of B in $\underline{M}$.

Then, a typical definition for $\underline{I}$ is

$$\underline{I} \equiv \int dt \int_{\Omega_t} d^2x \sqrt{\sigma} (\tilde{N} \underline{\tilde{\varepsilon}} - \tilde{V}^\alpha \underline{\tilde{j}}_\alpha), \tag{63}$$

where $\underline{\tilde{\varepsilon}}$ and $\underline{\tilde{j}}_\alpha$ are defined in the same way as usual except that this time they are evaluated for the surface Ω_t embedded in the reference spacetime. Thus, as noted, the choice of $\underline{I}$ effectively sets the zeros for the quasilocal quantities. As required by the formalism, $\underline{I}$ is defined entirely with respect to quantities defined over B.

Now, given T^α and $\underline{T}^\alpha$ observers watching $T^{*\alpha}$ and $\underline{T}^{*\alpha}$ observers, in general $\eta = u^{*\alpha} n_\alpha$ will not be equal to $\underline{\eta} = \underline{u}^{*\alpha} \underline{n}_\alpha$. Physically this means that in order for $(\Omega_t, \sigma_{\alpha\beta})$ to evolve in the same way in the two spacetimes, that surface will have to "move" at different speeds in each of the two. Then the transformation law for the quasilocal energy density with reference terms becomes

$$\varepsilon^* + \varepsilon^{m*} - \underline{\varepsilon}^* = \frac{1}{\lambda}(\varepsilon + \varepsilon^m) + \eta(\varepsilon^{\updownarrow} + \varepsilon^{m\updownarrow}) - \left(\frac{1}{\underline{\lambda}} \underline{\varepsilon} + \underline{\eta} \underline{\varepsilon}^{\updownarrow} \right). \tag{64}$$

Thus, even if the T^α observers are static and the $T^{*\alpha}$ observers move perpendicularly to Ω_t as discussed in the earlier example, the simple Lorentz transformation law will not hold, as the basic and reference components of the QLE will transform according to different perpendicular velocities. This situation is reminiscent of the relationship between spacetimes which differ from one another by conformal transformations; in general the quasilocal mass for a system of finite size will not be conformally invariant due to the lack of conformal invariance of the background action functional which specifies a reference background spacetime [29].

3.5 Thin shells - an operational definition of QLE

In this section we examine in some detail a correspondence between the quasilocal formalism and the mathematics describing thin shells in general relativity which was developed by Israel in 1966 [30]. This was noted in passing in [6] but here we shall examine it in more detail and use it to reinterpret the quasilocal energy from an operational point of view. We begin by reviewing the purely gravitational thin shell work, then add in matter fields, and finally consider the implications of this correspondence for the recently proposed AdS/CFT inspired reference terms.

3.5.1 Pure gravitational fields

Israel considered the conditions that two spacetimes, each of which has a boundary, must satisfy so that they may be joined along those boundaries and yet still satisfy Einstein's equations. He showed that as an absolute minimum the spacetimes must induce the same metric on the common boundary hypersurface. Further the Einstein equations will only be satisfied at the boundary if its extrinsic curvature in each of the two spacetimes is the same. If those curvatures are not the same then a singularity exists in the (joined) spacetime at the hypersurface. However that singularity is sufficiently mild that it may be accounted for by a stress energy tensor defined on that boundary. The change in curvature may then be interpreted as a manifestation of a thin shell of matter.

Modifying Israel's notation and sign conventions to be compatible with those used in this paper that stress tensor is defined as follows. Consider a spacetime $\mathcal{M}$ divided into two regions $\mathcal{V}^+$ and $\mathcal{V}^-$ by a timelike hypersurface B. Let the metric on $\mathcal{V}^+$ be $g^+_{\alpha\beta}$ and the metric on $\mathcal{V}^-$ be $g^-_{\alpha\beta}$, and assume that they induce the same metric $\gamma_{\alpha\beta}$ on B. Further, let $n^{+\alpha}$ and $n^{-\alpha}$ be the spacelike unit normals of B on each of its sides and define $\Theta^+_{\alpha\beta}$ and $\Theta^-_{\alpha\beta}$ to be the extrinsic curvature of B in $\mathcal{V}^+$ and $\mathcal{V}^-$ respectively. Then, Einstein's equation will only be satisfied if a thin shell of matter is present at B with stress-energy tensor $S_{\alpha\beta} = \frac{1}{\kappa}\left\{(\Theta^+_{\alpha\beta} - \Theta^+\gamma_{\alpha\beta}) - (\Theta^-_{\alpha\beta} - \Theta^-\gamma_{\alpha\beta})\right\}$. Note that this is the stress-energy tensor in the surface. To write it as a four dimensional stress energy tensor we must add in an appropriate Dirac delta function.

Now let Ω_t be a foliation of B generated by a timelike vector field $T^\alpha \equiv \tilde{N}\tilde{u}^\alpha + \tilde{V}^\alpha$ (which as usual lies entirely in the tangent space to B). Then observers who are static with respect to the foliation will observe the thin shell to have the following the energy, momentum, and stress densities:

$$\mathcal{E} = S_{\alpha\beta}\tilde{u}^\alpha\tilde{u}^\beta = \frac{1}{\kappa}\left\{\tilde{k}^+ - \tilde{k}^-\right\}, \tag{65}$$

$$\mathcal{J}_\alpha = S_{\gamma\delta}\sigma^\gamma_\alpha\tilde{u}^\delta = \frac{1}{\kappa}\left\{\sigma^\gamma_\alpha\tilde{u}^\delta\nabla_\gamma n^+_\delta - \sigma^\gamma_\alpha\tilde{u}^\delta\nabla_\gamma n^-_\delta\right\}, \text{ and} \tag{66}$$

$$\mathcal{S}_{\alpha\beta} = S_{\gamma\delta}\sigma^\gamma_\alpha\sigma^\delta_\beta = \frac{1}{\kappa}\left\{(\tilde{k}^+_{\alpha\beta} - (\tilde{k}^+ - n^{+\delta}\tilde{a}_\delta)\sigma_{\alpha\beta}) - (\tilde{k}^-_{\alpha\beta} - (\tilde{k}^- - n^{-\delta}\tilde{a}_\delta)\sigma_{\alpha\beta})\right\}, \tag{67}$$

where $\tilde{k}^\pm_{\alpha\beta} = -\sigma^\gamma_\alpha\sigma^\delta_\beta\nabla_\gamma n^\pm_\delta$ and $\tilde{k}^\pm = \sigma^{\alpha\beta}\tilde{k}^\pm_{\alpha\beta}$ are, as they were in previous sections, the extrinsic curvature of the surface Ω_t in a (local) foliation of $\mathcal{M}$ perpendicular to B. $\tilde{a}^\alpha$ retains its earlier meaning.

The correspondence between the quasilocal and thin shell formalisms is now obvious. Consider the surface $(B, \gamma_{\alpha\beta})$ embedded in a spacetime $(\mathcal{M}, g_{\alpha\beta})$ and a reference spacetime $(\underline{\mathcal{M}}, \underline{g}_{\alpha\beta})$. Further let $(\mathcal{M}, g_{\alpha\beta})$ be isomorphic to $(\mathcal{V}^+, g^+_{\alpha\beta})$ (or more properly the portion of $(\mathcal{M}, g_{\alpha\beta})$ to one side of B is isomorphic to $(\mathcal{V}^+, g^+_{\alpha\beta})$), and in the same sense let $(\underline{\mathcal{M}}, \underline{g}_{\alpha\beta})$ be isomorphic to $(\mathcal{V}^-, g^-_{\alpha\beta})$. Then for observers living on B and defining their notion of simultaneity according to the foliation Ω_t,

$$\mathcal{E} = \varepsilon - \underline{\varepsilon}, \tag{68}$$

$$\mathcal{J}_\alpha = j_\alpha - \underline{j}_\alpha, \text{ and} \tag{69}$$
$$\mathcal{S}_{\alpha\beta} = s_{\alpha\beta} - \underline{s}_{\alpha\beta}, \tag{70}$$

where $\underline{s}_{\alpha\beta}$ is defined in the obvious way.

We may interpret this mathematical identity of the formalisms in a couple of ways. First, as in [6] we may note that the quasilocal work formalism provides an alternate derivation of the thin shell junction conditions and stress energy tensor. Namely we may consider two quasilocal surfaces on either side of the shell and consider the limit as the two go to the shell. In that case any reference terms will match and cancel and we will be left with the stress energy tensor defined above. This derivation is quite different from the one used by Israel.

From a slightly different perspective we may view the thin shell work as providing an operational definition of the quasilocal energy. Given a reference spacetime $\underline{\mathcal{M}}$ which is assumed to have energy zero, then the QLE contained within a two surface Ω_t of a spacetime $\mathcal{M}$ can be defined as the energy of a shell of matter $\underline{\Omega}_t$ in $\underline{\mathcal{M}}$ that has the same intrinsic geometry as Ω_t (including the rate of change of those properties – see the previous section) and a stress energy tensor defined so that the spacetime outside of $\underline{\Omega}_t$ is identical to that outside of Ω_t in $\mathcal{M}$ while inside it remains $\underline{\mathcal{M}}$. In fact, the QLE with the embedding reference terms that we have considered is defined if and only if the fields outside of Ω_t can be replicated by a shell of stress energy with the same intrinsic geometry embedded in $\underline{\mathcal{M}}$. Provided that we can embed Ω_t in the reference spacetime $\underline{\mathcal{M}}$, by the construction that we have considered in this section we can define the relevant stress energy for a shell in $\underline{\mathcal{M}}$.

This operational interpretation of the QLE serves to highlight certain aspects of its definition. First, in the spirit of the Gauss law from electromagnetism the quasilocal approach assumes that a bulk property of a finite region of spacetime is fully reflected in the values of fields measured on the surface of that region. Just as the Gauss law of electromagnetism yields a measure of the electric charge contained within a surface from the components of the electric field perpendicular to it, here we attempt to define the energy contained in a region from properties of its boundaries. Therefore two different configurations of fields and matter that produce a given set of fields at a given surface will – by the very nature of the approach – be defined to have the same energy. This correspondence between the quasilocal and thin shell approaches strongly support this assumption.

Related to this is the fact that the quasilocal energy as defined by Brown and York (and extended here), is additive. That is, the total energy contained within regions Ω_a and Ω_b is equal to the energy contained within $\Omega_a \cup \Omega_b$. This is a very natural requirement for energy to meet and combined with a definition of the total amount of energy in a spacetime (such as the ADM energy) is basically equivalent to the "Gauss law" assumption that we noted in the previous paragraph. If we know the total energy in a spacetime and also can measure the amount of energy outside of a surface Ω_t then the additivity tells us how much energy is inside Ω_t and also requires that any configuration of matter and fields that produces the correct geometry at the surface Ω_t must have the same energy.

3.5.2 Including matter fields

So far we have only considered the purely gravitational case. With the inclusion of non-zero electromagnetic fields in the outer spacetime we must also include charge and current densities in the thin shell if the inner (reference) spacetime doesn't have such fields. These charge and current densities are defined to account for discontinuities in the electromagnetic field just as the stress tensor is defined to account for discontinuities in the gravitational field/geometry of spacetime. Their calculation is an exercise from undergraduate electromagnetism [31]. Specifically if there is no EM field within the shell then for the foliation Σ_t the electric charge density on the shell Ω_t is $\frac{2}{\kappa} n^\alpha \tilde{E}_\alpha$.

Given an EM potential A_α, observers on the surface of the shell who evolve via the vector field T^α will define a Coulomb potential $-T^\alpha A_\alpha = \tilde{N}\tilde{\Phi} - \hat{V}^\alpha \hat{A}_\alpha$. In the usual way we then define the energy of the charge density in the field as the charge times the potential. That is, $\frac{2}{\kappa} n^\alpha \tilde{E}_\alpha(-T^\alpha A_\alpha) = \tilde{N}\varepsilon^m - \tilde{V}^\alpha \tilde{\jmath}^m_\alpha$. As usual this component of the energy is gauge dependent.

The surface charges and currents do not change the definition of the surface stress energy tensor which was defined entirely by the Einstein equations. As such they also don't change the definitions of $\mathcal{E}$, $\mathcal{J}_\alpha$ and $\mathcal{S}_{\alpha\beta}$. Therefore, if we include the stress energy with the energy of the shell in the gauge field, the total energy density in a thin shell evolving by the vector field T^α is $\tilde{N}(\tilde{\varepsilon} + \tilde{\varepsilon}^m) - \tilde{V}^\alpha(\tilde{\jmath}_\alpha + \tilde{\jmath}^m_\alpha)$. This of course is exactly the same as the QLE of the region of space on and inside of the shell as measured by a set of observers being evolved by the same vector field, and so we see that the correspondence between thin shell and quasilocal energies remains.

The preceding reasoning may be repeated for the dilaton. The dilaton charge in a given volume is given by the integral of $n^\alpha \nabla_\alpha \phi$ over the surface enclosing that volume. For black hole solutions, the value of the dilaton charge is constrained by demanding the spacetime has no singularities on or outside of the outermost horizon [32]. In the thin shell case, $n^\alpha \nabla_\alpha \phi$ yields the dilaton charge density on the shell Ω_t if there is a constant dilaton field inside.

3.5.3 The AdS/CFT inspired reference terms

Recently there has been quite a bit of interest (references [20]) in defining the reference terms with respect to intrinsic quantities of the boundaries rather than their extrinsic curvature when embedded in a reference spacetime. These new reference terms have been inspired by the AdS/CFT correspondence and there is much to say for them. In the first place apart from the asymptotically flat cases it isn't really obvious what spacetime should be used as a reference. In the second, even after that choice has been made it is in general either computationally difficult or actually impossible to embed a given quasilocal surface in a given reference spacetime (as an example see [13] where Martinez tries to embed the an $r =$ constant $t =$ constant surface from Kerr space into flat space). Intrinsic reference terms remove this problem and as such are of great interest.

Unfortunately however, the correspondence between the thin shells and QLE does not (in general) hold for the intrinsic reference terms proposed so far. For example consider the flat space reference used in some of these papers. There $\underline{\varepsilon} = \frac{2}{\kappa}\sqrt{R^{(2)}}$, where $R^{(2)}$ is the Ricci scalar for Ω_t. Recalling some elementary differential geometry, the extrinsic curvature of a two surface in flat three space is $\underline{k} = \frac{1}{X_1} + \frac{1}{X_2}$ while the intrinsic curvature is $R^{(2)} = \frac{1}{X_1 X_2}$, where X_1 and X_2 are the principal radii of curvature for the surface. Then, $\underline{k} \geq 2\sqrt{R^{(2)}}$ and the equality only holds if $X_1 = X_2$. That is the two reference energies are only exactly equal (and so the thin shell correspondence only holds) when Ω_t is a sphere. However, as Lau pointed out [20] the equality will also hold in the limit of an arbitrarily large Ω_t in which case the difference between the two terms goes to zero. Thus, in the cases where we can compare to thin shell calculations the quasilocal energy with this intrinsic reference term only matches for spherical shells or in the asymptotic limit.

Perhaps even more seriously the above observation implies that the energy contained by a finite ellipsoidal surface is non-zero in flat space while the energy contained by a similar sphere is zero. Given these observations it is probably best to follow Lau and view the these new reference terms as a short-cut for calculating the embedding reference terms in the asymptotic limit rather than as a replacement for them.

4 Examples

We now apply the foregoing work to a couple of examples. In the first we investigate the quasilocal energy distribution in a Reissner-Nordström (RN) spacetime. In the course of this investigation we use the thin shell analogy to show that the quasilocal energy reduces to the intuitive Newtonian limit. For the Hamiltonian we also explicitly illustrate our earlier comments on gauge (in)dependence. Finally, in the last example we will come to grips with naked black holes.

4.1 Reissner-Nordström spacetimes

In this section we study the distribution of energy in a RN spacetime. The dyonic solution has metric

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + r^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{71}$$

where $F(r) \equiv 1 - \frac{2m}{r} + \frac{E_0^2+G_0^2}{r^2}$. m is the mass, and E_0 and G_0 are respectively the electric and magnetic charges of the hole. The accompanying electromagnetic field is described by

$$F = -\frac{E_0}{r^2}dt \wedge dr + G_0 \sin\theta d\theta \wedge d\varphi. \tag{72}$$

A local vector potential generating this field is

$$A = -\frac{E_0}{r}dt - G_0 \cos\theta d\varphi + d\chi, \tag{73}$$

where $\chi = \chi(t, r, \theta, \varphi)$ is any function over M. Note that the potential given above with $\chi = 0$ is not defined over all of M since $d\varphi$ is not defined at $\theta = 0, \pi$. In section 3.3 we saw that our Lagrangian formalism as constituted is not suitable for discussing dyonic spacetimes. As such we focus on electric black holes in the following subsections. The results for magnetic black holes are identical if we switch E_0 and G_0 in the following and add in the appropriate $\star$'s.

Setting $G_0 = 0$ we calculate the quasilocal quantities measured by static, spherically symmetric observers. Specifically, we consider a surface of observers Ω_0 defined as the intersection of the $t = t_0$ surfaces $r = r_0$ surfaces (where t_0 and r_0 are constants). They are evolved by the vector field $T^\alpha = N(r)\tilde{u}^\alpha$. $N(r)$ is the lapse function while the shift $V^\alpha = 0$. For any choice of $N(r)$ the observers will be static in the sense that they don't observe any changes in the spatial metric; the lapse just determines how they choose to measure their time. In particular, choosing $N(r) = \sqrt{F(r)}$ the observers measure time according to the coordinate t, while choosing $N(r) = 1$ the observers measure time in the "natural" way (that is $T^\alpha T_\alpha = -1$).

Then, $\tilde{u}^\alpha = \frac{1}{\sqrt{F(r)}}\partial_t^\alpha$, $n^\alpha = \sqrt{F(r)}\partial_r^\alpha$, and a series of straightforward calculations yields

$$\varepsilon = -\frac{1}{4\pi r^2}\sqrt{r^2 F}, \tag{74}$$

$$\varepsilon^m = \frac{1}{4\pi r^2}\frac{E_0(E_0 - r\partial_t\chi)}{\sqrt{r^2 F}}, \text{ and} \tag{75}$$

$$\underline{\varepsilon} = -\frac{1}{4\pi r}. \tag{76}$$

$$\tag{77}$$

We have made the substitution $\kappa = 8\pi$ and so henceforth work in geometric units where $G = c = 1$. The angular momentum terms are all zero as we would expect for these non-rotating spacetimes.

4.1.1 Static geometric energy

Here we calculate the quasilocal energy associated with the density ε. We label it the geometric energy since it depends only on the extrinsic curvatures. Then, not including the reference term

$$E_{Geo} = \int_{\Omega_0} d^2x\sqrt{\sigma}\varepsilon = -\sqrt{r^2 - 2mr + E_0^2}. \tag{78}$$

In the large r limit this becomes $E_{Geo} = -r + m + \frac{1}{2r}(m^2 - E_0^2)$. The $\underline{\varepsilon}$ reference term is

$$\underline{E} = \int_{\Omega_0} d^2x \sqrt{\sigma}\underline{\varepsilon} = -r, \tag{79}$$

so

$$E_{Geo} - \underline{E} = r - \sqrt{r^2 - 2mr + E_0^2} \approx m + \frac{1}{2r}(m^2 - E_0^2), \tag{80}$$

in the large r limit.

This is what we would expect from an application of Newtonian intuition to the (equivalent) thin shell situation. For this viewpoint consider how much energy it would take to construct a shell of radius r with mass m and charge E_0 from material residing "out at infinity". Using Newton's and Coulomb's laws, it is straightforward to show that $-\frac{1}{2r}(m^2 - E_0^2)$ units of work are required to assemble the shell. It is then very natural to say that this energy is "stored" in the field, outside radius r. Now, an observer sitting far from the hole and measuring how a neutral particle is accelerated by the gravitational field would say that the total mass contained in the spacetime is m. Then assuming that the energy is additive, the energy contained on and/or inside the shell with radius r is

$$(\text{total energy}) - (\text{energy in fields outside of r}) = m + \frac{1}{2r}(m^2 - E_0^2), \tag{81}$$

as we calculated above. An alternative way of looking at this is to say that m units of energy were used to create the mass m at infinity and then $\frac{1}{2}(m^2 - E_0^2)$ units of energy were stored in the fields outside of the shell. Assuming that conservation of energy holds once the matter has been created then we again obtain the above result once the shell has been constructed. In either case, our calculation reduces properly in the Newtonian limit. This limiting case was first considered in the original Brown and York paper [6].

Returning to the full expression we note that $E_{Geo} - \underline{E}$ monotonically decreases as r increases, reaching a minimum of m at infinity. Thus the energy contained in the fields is negative, which is what we would expect for a binding energy.

Finally note that for an extreme black hole where $|E_0| = m$, $E_{Geo} - \underline{E} = m$ is a constant. From the Newtonian shell point of view this makes sense. If we consider the construction of a shell with charge $|E_0| = m$ and mass m out of particles which also have equal mass and charge, then equal but opposite electric and gravitational forces would act on the particles during the construction. Thus, no work must be done to build the shell and so no energy is stored on the fields. Alternately equal amounts of positive and negative energy are stored in the electric and gravitational fields and cancel each other out. The only energy is then that stored in the mass.

We may think of this geometric energy as a "configuration energy" that arises from the spatial relationships of different parts of the spacetime to each other. By contrast in the next section where we include the gauge dependent terms we will see that our expression for the energy also includes "position" terms that arise due to the position of the different parts of the spacetime in the gauge potential. Of course, the form of the gauge potential is determined up to a gauge transformation by the matter so this view of the terms as being configurational versus positional is at best a rough way to think of them.

4.1.2 Static total energy

Next consider $\varepsilon + \varepsilon^m$, the full energy density that was derived from the variational calculations. Then

$$E_{tot} = \int_{\Omega_0} d^2x \sqrt{\sigma}(\varepsilon + \varepsilon^m) = \frac{-r^2 + 2mr - E_0 r \partial_t \chi}{\sqrt{r^2 - 2mr + E_0^2}}. \tag{82}$$

As we have emphasized before this expression is manifestly gauge dependent. Even worse however is the fact that this energy will in general diverge at the outer horizon of a black hole. Before we deal with that worry however, let us consider the usual $r \to \infty$ limit.

If we demand that A_α has the same spherical and time translation symmetries as the spacetime, then $\chi = -\Phi_\infty t + f(r)$ where $\Phi_\infty \equiv \lim_{r\to\infty} \Phi$ is a constant and $f(r)$ is an arbitrary function of r. Then,

$$E_{tot} - \underline{E} = r - \frac{r^2 - 2mr - E_0 r \partial_t \chi}{\sqrt{r^2 - 2mr + E_0^2}} \approx (m + E_0 \Phi_\infty) + \frac{1}{2r}(m^2 + E0^2 + 2mE_0\Phi_\infty). \tag{83}$$

Since the total energy is the sum of the geometric energy and the gauge dependent term, it isn't surprising that this Newtonian limit is the sum of the Newtonian limit of the geometric energy and the "positional" potential term. We can think of Φ_∞ as the zero level of the potential throughout space (it remains even if $E_0 \to 0$) and so by the thin shell analogy can think of $E_0\Phi_\infty$ as the energy cost for creating charge in this spacetime before bringing it in from infinity. For extreme black holes recall that $E_{Geo} - \underline{E} = m$ so the $E_{tot} - \underline{E} = m + \int_{\Omega_t} d^2x \sqrt{\sigma} \varepsilon^m$ and the only energy is the mass m plus the energy of the charge with respect to the potential.

In most situations the exact choice of gauge is just a matter of convenience. For black hole spacetimes however, we noted above that for most gauge choices E_{tot} will diverge on the horizon. This divergence can be directly traced to the fact that the Coulomb potential $\Phi = -u^\alpha A_\alpha = \frac{1}{\sqrt{F}}(\frac{E_0}{r} - \partial_t \chi)$ also diverges at the horizon. To remove both divergences we must choose χ such that $\partial_t \chi \to \frac{E_0}{r_+}$ as $r \to r_+$ where r_+ is the outer black hole horizon. That is we set the Coulomb potential to zero on the black hole horizon. Assuming that A_α has the symmetries that we discussed above we must then choose $\Phi_\infty = -\frac{E_0}{r_+}$. Making that choice, after a little algebra we obtain

$$E_{tot} = -r\sqrt{\frac{r - r_+}{r - r_-}} \tag{84}$$

where $r_\pm = m \pm \sqrt{m^2 - E_0^2}$ are the radial positions of inner and outer horizons. This gauge will also be used for the naked black holes. For extreme black holes $r_+ = r_- = |E_0| = m$ and so $E_{tot} = \underline{E}$ everywhere. Physically we have chosen the gauge so that the potential energy is a constant and everywhere equal $-m$. The (in this case negative) electric potential energy cancels the mass-energy while at the same time the positive energy of the electric field cancels the negative binding energy of the gravitational field.

Reflecting on this section we see that the total energy may in a very real sense be split into two parts. In the last section we saw that the geometric part depends only on the configuration of the spacetime. Examining the Newtonian limit we saw that it appears to include not only the gravitational but also the electromagnetic "configurational" energies. By contrast in this section we saw that the gauge dependent part exclusively deals with the potential of the matter relative to the gauge field. As we have seen, for a given solution to the Einstein-Maxwell equations the total QLE for a given surface may take any value (including zero) depending on the exact gauge choice. As such it is clear that this gauge dependent part of the energy is not reflected in the stress-energy tensor. On the other hand we should not then conclude that this gauge dependent part is entirely meaningless. It certainly plays a role equal to the geometric energy in both in thermodynamics [7, 9] and in black hole pair creation [12].

4.1.3 Energies seen by radially moving observers

We next consider the energies measured by spherically symmetric sets of observers who are moving radially in the RN spacetime. As before we take Ω_t to be $r =$ constant, $t =$ constant surfaces but now take $T^{*\alpha} = N^* u^{*\alpha}$ where $u^{*\alpha} = \frac{1}{\lambda} u^\alpha + \eta \tilde{n}^\alpha = \gamma(u^\alpha + v_\vdash \tilde{n}^\alpha)$. As in section 3.4, $v_\vdash$ is the speed of the $T^{*\alpha}$ observers in the $\tilde{n}^\alpha = \sqrt{F}\partial_r^\alpha$ direction as measured by a second set of observers evolving by $u^\alpha = \frac{1}{\sqrt{F}}\partial_t^\alpha$.

Then, a straightforward calculation shows $\varepsilon^{\updownarrow} = 0$ so

$$E^*_{Geo} = \gamma E_{Geo} = -\gamma r\sqrt{F}. \tag{85}$$

Unfortunately, from the point of view of simplicity, within our gauge freedom $\varepsilon^{m\updownarrow}$ is not necessarily zero. Even if we restrict ourselves to gauge choices that give A_α the same symmetries as the spacetime, we have $\chi = -\Phi_\infty t + f(r)$ where as noted earlier Φ_∞ is a constant and f is an arbitrary function. Then

$$\varepsilon^{m\updownarrow} = -\frac{E_0}{4\pi}\sqrt{F}\partial_r f. \tag{86}$$

In the interests of simplicity however, we make the standard gauge choice for electrostatics and let $\partial_r f = 0$. Then the Lorentz-type transformation laws will apply and

$$E^*_{tot} = \gamma E_{tot} = -\gamma r\left(\sqrt{F} - \frac{E_0}{r\sqrt{F}}\left[\Phi_\infty + \frac{E_0}{r}\right]\right). \tag{87}$$

As before we must choose $\Phi_\infty = -\frac{E_0}{r_+}$ if we don't want this quantity to diverge at the horizon.

To include the reference terms, we must first find a time vector $\underline{T}^{*\alpha}$ for the reference spacetime such that $\underline{T}^{*\alpha}\underline{T}^*_\alpha = T^{*\alpha}T^*_\alpha$ and $\pounds_{\underline{T}^*}\underline{\sigma}_{\alpha\beta} = \pounds_{T^*}\sigma_{\alpha\beta}$. This prescription gives

$$\underline{T}^{*\alpha} = \gamma\left(\sqrt{1-(1-F)v_\vdash^2}\,\underline{u}^\alpha + v_\vdash\sqrt{F}\underline{\tilde{n}}^\alpha\right), \tag{88}$$

where $\underline{u}^\alpha = \partial^\alpha_{\underline{t}}$, $\underline{\tilde{n}}^\alpha = \partial^\alpha_{\underline{r}}$ and $\underline{t}$ and $\underline{r}$ the usual time and radial coordinates for Minkowski space. Then

$$\underline{v}_\vdash \equiv -\frac{\underline{T}^{*\alpha}\underline{\tilde{n}}_\alpha}{\underline{T}^{*\alpha}\underline{u}_\alpha} = \frac{v_\vdash\sqrt{F}}{\sqrt{1-(1-F)v_\vdash^2}} \quad \text{and} \quad \underline{\gamma} = \gamma\sqrt{1-(1-F)v_\vdash^2}. \tag{89}$$

Thus

$$\underline{E}^* = \underline{\gamma}\underline{E} = -r\gamma\sqrt{1-(1-F)v_\vdash^2}, \tag{90}$$

and we have

$$E^*_{Geo} - \underline{E}^* = r\gamma\left(\sqrt{1-(1-F)v_\vdash^2} - \sqrt{F}\right), \tag{91}$$

and

$$E^*_{tot} - \underline{E}^* = r\gamma\left(\sqrt{1-(1-F)v_\vdash^2} - \left(\sqrt{F} - \frac{E_0}{r\sqrt{F}}\left[\Phi_\infty + \frac{E_0}{r}\right]\right)\right). \tag{92}$$

As they stand these expressions are quite complicated and their physical interpretation isn't at all obvious. Thus let us consider the large r/small v limit. To first order in $\frac{1}{r}$ and first order in v^2

$$E^*_{Geo} - \underline{E}^* \approx m + \frac{1}{2r}\left(m^2 - E_0^2\right) - \frac{1}{2}mv_\vdash^2, \tag{93}$$

and

$$E^*_{tot} - \underline{E}^* \approx (m + E_0\Phi_\infty) - \frac{1}{2}(m + E_0\Phi_\infty)v_\vdash^2 + \frac{1}{2r}(E_0^2 + m^2 + 2mE_0\Phi_\infty). \tag{94}$$

These results are quite interesting. We see that the motion of the observers actually serves to decrease the quasilocal energy that they measure. Specifically we see that both the total and geometric boosted quasilocal energies are equal to their unboosted counterparts minus a kinetic term equal to the $\frac{1}{2}$(Total Energy of Fields)$v_\vdash^2$. This effect, also noted in [24], is in stark contrast to both of the no-reference-term quantities whose measure increases with motion. Some discussion of why this happens may be found in section 4.2.2 where we consider the equivalent effect for naked black holes.

Next, in preparation for studying those naked black holes, we consider the specific case in which a spherically symmetric group of observers are falling into a RN black hole. These observers are assumed to have started with velocity close zero "close to infinity" and then fallen along a radial geodesic inwards towards the black hole. Rigorously the geodesic is the one that with respect to the standard time foliation has radial velocity zero at infinity and 1 at the outer horizon. Now, a test particle starting with velocity zero at radial coordinate r_0 and then allowed to fall towards a black hole on a radial geodesic will have coordinate velocity

$$\frac{dr}{d\tau} = -\sqrt{F(r_0) - F(r)} \tag{95}$$

as a function of r, where τ is the proper time. Thus an observer infalling on a geodesic that was static at infinity will have coordinate velocity $\frac{dr}{d\tau} = -\sqrt{1 - F(r)}$.

Let these observers measure time in the natural way (that is $\tilde{N} = 1$), $T^{*\alpha} = \frac{1}{\sqrt{F}} u^\alpha - \sqrt{\frac{1-F}{F}} \tilde{n}^\alpha$. Then the instantaneous radial velocity of the $T^{*\alpha}$ observers as measured in the static u^α frame is

$$v_\vdash \equiv -\frac{T^{*\alpha}\tilde{n}_\alpha}{T^{*\beta}u_\beta} = -\sqrt{1-F}, \tag{96}$$

and so the Lorentz factor is $\gamma = \frac{1}{\sqrt{F}}$.

Substituting this value for γ into equations (85,87,90) and making the gauge choice $\Phi_\infty = -\frac{E_0}{r_+}$ so that E^*_{tot} doesn't diverge at the horizon,

$$\begin{aligned} E^*_{Geo} &= -r, && (97) \\ E^*_{tot} &= -\frac{r^2}{r - r_-}, \text{ and} && (98) \\ \underline{E}^* &= -r\sqrt{2-F}. && (99) \end{aligned}$$

Note that as $r \to r_+$ all of these take non-zero values. By contrast E_{Geo} and E_{tot} both are zero at r_+. Also, keep in mind that for a near extreme black hole, $r_+ \approx r_-$. Therefore for a black hole that is very close to being extreme, the observers will measure E^*_{tot} to have a very large negative value as they approach the horizon.

Including the reference terms,

$$E^*_{Geo} - \underline{E}^* = r(\sqrt{2-F} - 1), \tag{100}$$

and

$$E^*_{tot} - \underline{E}^* = r\left(\sqrt{2-F} - \frac{r}{r - r_-}\right) \tag{101}$$

So near the horizon the infalling gravitational energy (including the reference term) goes to $(\sqrt{2} - 1)r_+$ compared to r_+ for the static gravitational energy. By contrast, the infalling total energy (including reference term) attains arbitrarily large negative values as the observers approach the horizon for black holes that are arbitrarily close to being extreme. Static observers however, will measure $E_{tot} - \underline{E} = r_+$ as they hover around the horizon. The difference is essentially due to the hugely boosted matter terms. As we have seen the boosting of the geometric terms has a comparatively minor effect.

4.1.4 The Hamiltonian

Finally we return to static sets of observers measuring time according to the time coordinate t (that is lapse $N = \sqrt{F}$) and calculate the Hamiltonian energy measured by these observers. As usual the surfaces of observers are spherical $t =$ constant and $r =$ constant surfaces. Then

$$
\begin{aligned}
H_{Geo} &= NE_{Geo} = -rF, && (102)\\
H_{Geo} - \underline{H} &= N(E_{Geo} - \underline{E}) = \sqrt{r^2F}(1-\sqrt{F}), && (103)\\
H_{tot} &= NE_{tot} = -r + 2m + E_0\partial_t\chi, \qquad \text{and} && (104)\\
H_{tot} - \underline{H} &= N(E_{tot} - \underline{E}) = 2m + E_0\partial_t\chi + \sqrt{r^2F} - r && (105)
\end{aligned}
$$

In the large r limit, $H_{Geo} - \underline{H} \approx m - \frac{m^2+E_0^2}{2r}$ and $H_{tot} - \underline{H} \approx (m + E_0\Phi_\infty) - \frac{m^2-E_0^2}{2r}$. Thus as is usual for asymptotically flat spacetimes the Hamiltonian corresponds to the QLE in the $r \to \infty$ limit. Note too though that in the large r limit these Hamiltonian's don't agree with the Newtonian limits that we discussed earlier.

Finally let us illustrate the earlier comments on gauge invariance. To avoid the complications of singularities we redefine M to be the region of $\mathcal{M}$ contained by the two timelike hypersurfaces $r = r_1$ and $r = r_2$ where $r_+ < r_1 < r_2$. Again we foliate that region according to the standard time coordinate t. Since we are only interested in the gauge invariance of the Hamiltonian and the reference terms are manifestly gauge invariant we ignore them for this calculation.

Then the total Hamiltonian for the action I_{EMdil} for the region M is

$$H_M = \Sigma H_{tot} = (r_1 - r_2) + \frac{E_0}{2}[\partial_t\chi]_{r1}^{r2} \tag{106}$$

where $[\partial_t\chi]_{r_1}^{r_2} = \partial_t\chi|_{r=r_2} - \partial_t\chi|_{r=r_1}$. The sum is over the two boundary components. Consider the gauge dependence of this Hamiltonian. In section 3.2 we saw that we could only expect it to be gauge independent if M is was a region containing no singularities and $\pounds_T\bar{A}_\alpha = 0$. Well, there are no singularities in M and a quick calculation shows that $\pounds_T\bar{A}_\alpha = 0$ implies that $\partial_t\chi$ is constant over Σ_t. If this is true then $[\partial_t\chi]_{r_1}^{r_2} = 0$ and the Hamiltonian is gauge independent as we expect.

4.2 Naked black holes

Finally, we consider naked black holes. They are low energy limit solutions to string theory and are characterized by the fact that static observers hovering close to their horizons see only very small curvatures while infalling observers are crushed by very large tidal forces. They are naked in the sense that even though they are not Planck scale themselves, Planck scale curvatures may still be observed outside of their horizons. Several classes of these holes were studied in a couple of papers by Horowitz and Ross [25] but here we will consider only those satisfying the equations of motion (6–9). The naked black holes are then a subset of the following class of Maxwell-Dilaton black hole solutions. The metric is given by

$$ds^2 = -F(r)dt^2 + \frac{dr^2}{F(r)} + R(r)^2(d\theta^2 + \sin^2\theta d\varphi^2), \tag{107}$$

where

$$F(r) = \frac{(r-r_+)(r-r_-)}{R^2} \quad \text{and} \quad R(r) = r\left(1 - \frac{r_-}{r}\right)^{a^2/(1+a^2)}, \tag{108}$$

and r_+ and r_- are respectively the inner and outer horizons of the black hole. The accompanying dilaton and electromagnetic fields are defined by

$$e^{-2\phi} = \left(1 - \frac{r_-}{r}\right)^{2a/(1+a^2)} \tag{109}$$

and

$$\star F = \frac{G_0}{r^2} dt \wedge dr. \tag{110}$$

These solutions are all magnetic black holes so as we have discussed earlier we work with the dual electromagnetic field tensor. The ADM mass and magnetic charge are

$$\begin{aligned} M &= \frac{r_+}{2} + \frac{1-a^2}{1+a^2}\frac{r_-}{2} \text{ and} & (111) \\ G_0 &= \left(\frac{r_+ r_-}{1+a^2}\right)^{1/2}. & (112) \end{aligned}$$

Solving this pair of equations in terms of r_+ and r_- we find $r_\pm = \frac{1\mp a^2}{1-a^2}(M \pm \sqrt{M^2-(1-a^2)G_0^2})$ for $a \neq 1$ or $r_+ = 2M$ and $r_- = G_0^2/M$ for $a = 1$. Note that when $a = 0$ these expressions reduce to those for a magnetically charged RN black hole.

Massive near-extreme members of this class of solutions are naked. To see this we note that in terms of the orthonormal tetrad $\{u^\alpha, \tilde{n}^\alpha, \hat{\theta}^\alpha, \hat{\phi}^\alpha\}$ where $u^\alpha = 1/\sqrt{F}\partial_t^\alpha$, $\tilde{n}^\alpha = \sqrt{F}\partial_r^\alpha$, $\hat{\theta}^\alpha = 1/R\partial_\theta^\alpha$ and $\hat{\varphi}^\alpha = 1/(R\sin\theta)\partial_\varphi^\alpha$ the non-zero components of the Riemann tensor are

$$\begin{aligned} \mathcal{R}_{u\tilde{n}u\tilde{n}} &= \frac{\ddot{F}}{2}, & (113) \\ \mathcal{R}_{\hat{\varphi}\hat{\theta}\hat{\varphi}\hat{\theta}} &= \frac{1-F\dot{R}^2}{R^2}, & (114) \\ \mathcal{R}_{u\hat{\theta}u\hat{\theta}} &= \mathcal{R}_{u\hat{\varphi}u\hat{\varphi}} = \frac{\dot{F}\dot{R}}{2R}, \text{ and} & (115) \\ \mathcal{R}_{\tilde{n}\hat{\theta}\tilde{n}\hat{\theta}} &= \mathcal{R}_{\tilde{n}\hat{\varphi}\tilde{n}\hat{\varphi}} = -\frac{\dot{F}\dot{R}}{2R} - \frac{F\ddot{R}}{R}. & (116) \end{aligned}$$

In the above and the following, overdots indicate partial derivatives with respect to r.

In the alternate moving tetrad $\{\tilde{u}^\alpha, n^\alpha, \hat{\theta}^\alpha, \hat{\varphi}^\alpha\}$ where we recall that $\tilde{u}^\alpha = \frac{1}{\lambda}u^\alpha - \eta\tilde{n}^\alpha$ and $n^\alpha = \frac{1}{\lambda}\tilde{n} - \eta u^\alpha$ the non-zero components of the Riemann tensor are (in terms of the non-moving components)

$$\begin{aligned} \mathcal{R}_{\tilde{u}n\tilde{u}n} &= \mathcal{R}_{u\tilde{n}u\tilde{n}} = \frac{\ddot{F}}{2}, & (117) \\ \mathcal{R}_{\tilde{u}\hat{\varphi}\tilde{u}\hat{\varphi}} &= \mathcal{R}_{u\hat{\varphi}u\hat{\varphi}} + \eta^2\left(\mathcal{R}_{u\hat{\varphi}u\hat{\varphi}} + \mathcal{R}_{\tilde{n}\hat{\varphi}\tilde{n}\hat{\varphi}}\right) = \frac{\dot{F}\dot{R}}{2R} - \eta^2\frac{F\ddot{R}}{R}, \text{ and} & (118) \\ \mathcal{R}_{n\hat{\varphi}n\hat{\varphi}} &= \mathcal{R}_{\tilde{n}\hat{\varphi}\tilde{n}\hat{\varphi}} + \eta^2\left(\mathcal{R}_{u\hat{\varphi}u\hat{\varphi}} + \mathcal{R}_{\tilde{n}\hat{\varphi}\tilde{n}\hat{\varphi}}\right) = -\frac{\dot{F}\dot{R}}{2R} - \frac{F\ddot{R}}{R} - \eta^2\frac{F\ddot{R}}{R}. & (119) \end{aligned}$$

$\mathcal{R}_{\hat{\varphi}\hat{\theta}\hat{\varphi}\hat{\theta}}$ is unchanged and we still have $\mathcal{R}_{\tilde{u}\hat{\theta}\tilde{u}\hat{\theta}} = \mathcal{R}_{\tilde{u}\hat{\varphi}\tilde{u}\hat{\varphi}}$ and $\mathcal{R}_{n\hat{\theta}n\hat{\theta}} = \mathcal{R}_{n\hat{\varphi}n\hat{\varphi}}$. Clearly if $a = 0$ then $R(r) = r$ and all of the components are the same as for the unboosted frame.

Now if $a \neq 0$ and we define $\delta = (1 - r_-/r_+)^{1/(1+a^2)}$, then the naked black holes are the subset of the above set whose parameters satisfy the conditions $\frac{\delta^2}{a^2} \ll \frac{1}{R_+^2} \ll 1$, where $R_+ = R(r_+)$. That is $\frac{a}{\delta} \gg R_+$ which in turn is much larger than the Planck length. We note that if $\delta \ll 1$ then $r_- \approx r_+$ and if $R_+ \gg 1$ then $M, G_0 \gg 1$. Thus naked holes are near-extreme as well as being very large (relative to the Planck length).

In the static frame as $r \to r_+$,

$$\begin{aligned} |\mathcal{R}_{u\tilde{n}u\tilde{n}}| &\to \frac{1}{R_+^2}\left(1 - \frac{2r_-}{(1+a^2)r_+}\right) \ll 1, & (120) \\ \mathcal{R}_{\hat{\varphi}\hat{\theta}\hat{\varphi}\hat{\theta}} &\to \frac{1}{R_+^2} \ll 1, & (121) \end{aligned}$$

$$\mathcal{R}_{u\hat{\varphi}u\hat{\varphi}} \rightarrow \frac{1}{2R_+^2}\left(1 - \frac{r_-}{(1+a^2)r_+}\right) \ll 1, \text{ and} \tag{122}$$

$$|\mathcal{R}_{\tilde{n}\hat{\varphi}\tilde{n}\hat{\varphi}}| \rightarrow \frac{1}{2R_+^2}\left(1 - \frac{r_-}{(1+a^2)r_+}\right) \ll 1. \tag{123}$$

Thus, all of the curvature components (and consequently the curvature invariants calculated from them) are small compared to the Planck scale.

By contrast if we fix the tetrad to be that carried by the infalling observers considered in the previous example, $\eta^2 = \gamma^2 v_\vdash^2 = \frac{1-F}{F}$ and as $r \rightarrow r_+$,

$$|\mathcal{R}_{\tilde{u}\hat{\varphi}\tilde{u}\hat{\varphi}}| \rightarrow \frac{a^2}{(1+a^2)^2}\frac{r_-^2}{r_+^2}\frac{1}{R_+^2\delta^2} \gg 1 \text{ and} \tag{124}$$

$$|\mathcal{R}_{n\hat{\varphi}n\hat{\varphi}}| \rightarrow \frac{a^2}{(1+a^2)^2}\frac{r_-^2}{r_+^2}\frac{1}{R_+^2\delta^2} \gg 1. \tag{125}$$

Thus these infalling observers see Planck scale curvatures. Interpreting these components in terms of the relative acceleration of neighbouring geodesics we see that the observers are laterally crushed by huge tidal forces.

4.2.1 QLE of naked black holes

If we consider a spherical shell of observers falling into a naked black hole, we would expect these tidal forces to cause the area of the shell to shrink at a very rapid rate. Now rates of change of area factor largely in the formalism that we have developed in this paper. In particular $\varepsilon^\updownarrow$ is, up to a normalization factor, exactly the (local) rate of change of the area of an infalling surface of observers. As such it is of interest to calculate the quasilocal energies measured by static versus infalling observers and to see how they compare to the observed curvatures. After a considerable amount of algebra we obtain the pleasantly simple

$$\varepsilon = -\frac{\dot{R}}{4\pi R^2}\sqrt{(r-r_+)(r-r_-)}, \tag{126}$$

$$\varepsilon + \varepsilon^m = -\frac{1}{4\pi R}\sqrt{\frac{r-r_+}{r-r_-}}, \tag{127}$$

$$\varepsilon^\updownarrow = \varepsilon^{m\updownarrow} = 0, \text{ and} \tag{128}$$

$$\underline{\varepsilon} = -\frac{1}{4\pi R}. \tag{129}$$

The gauge choice for the matter term is the same one that we used earlier. That is, we have chosen the gauge so that $\star A_\alpha \parallel u_\alpha$, as well as being static, spherically symmetric, and non-diverging on the black hole horizon. Though this is a long list of requirements, as noted earlier it is really little more than asserting that we make the standard gauge choice of electrostatics (or in this case magnetostatics).

For the infalling observers we again have $T^{*\alpha} = \frac{1}{\sqrt{F}}u^\alpha - \sqrt{\frac{1-F}{F}}\tilde{n}^\alpha$ which implies $v_\vdash = -\sqrt{1-F}$ and $\gamma = 1/\sqrt{F}$. By contrast the joint requirements that $\underline{T}^{*\alpha}\underline{T}^*_\alpha = T^{*\alpha}T^*_\alpha$ and $\pounds_{\underline{T}^*}\underline{\sigma}_{\alpha\beta} = \pounds_{T^*}\sigma_{\alpha\beta}$ give us

$$\underline{T}^{\star\alpha} = \sqrt{1+\dot{R}^2(1-F)}\underline{u}^\alpha - \dot{R}\sqrt{1-F}\underline{\tilde{n}}^\alpha, \tag{130}$$

$$\underline{v}_\vdash = -\frac{\dot{R}\sqrt{1-F}}{\sqrt{1+\dot{R}^2(1-F)}}, \text{ and} \tag{131}$$

$$\underline{\gamma} = \sqrt{1+\dot{R}^2(1-F)}. \tag{132}$$

Quantity	$\delta >\approx a^2$ $r \to r_+$	$\delta \ll a^2$ $r \to r_+$	$r \to \infty$
$-E_{Geo}$	0	0	r
$-E^*_{Geo}$	$R_+ \gg 1$	$\frac{a^2}{1+a^2}\frac{R_+}{\delta} \gg\gg 1$	r
$E_{Geo} - \underline{E}$	$R_+ \gg 1$	$R_+ \gg 1$	M
$E^*_{Geo} - \underline{E}^*$	$C_1 R_+ \gg 1$	$\frac{1+a^2}{2a^2} R_+ \delta \ll 1$	M
$-E_{tot}$	0	0	r
$-E^*_{tot}$	$\frac{R_+}{\delta} \gg\gg 1$	$\frac{R_+}{\delta} \gg\gg 1$	r
$E_{tot} - \underline{E}$	$R_+ \gg 1$	$R_+ \gg 1$	$0 < R_+\delta \ll 1$
$-(E^*_{tot} - \underline{E}^*)$	$\frac{R_+}{\delta} \gg\gg 1$	$\frac{R_+}{\delta} \gg\gg 1$	$-1 \ll R_+\delta < 0$

Figure 2: Asymptotic and near horizon behaviour of the quasilocal energies for near-extreme dilaton-Maxwell black holes. We have $\delta = (1 - r_-/r_+)^{1/(1+a^2)} \ll 1$, $R_+ = r_+\delta^{a^2} \gg 1$ and $R_+^2\delta^2 \ll 1$, where $R_+ = R(r_+)$. C_1 is a constant on the same order as 1.

Then, we have

$$E_{Geo} = -\sqrt{(r-r_+)(r-r_-)}\dot{R}, \tag{133}$$
$$E^*_{Geo} = -R\dot{R}, \tag{134}$$
$$E_{tot} = -R\sqrt{\frac{r-r_+}{r-r_-}}, \tag{135}$$
$$E^*_{tot} = -\frac{R^2}{r-r_-} \tag{136}$$
$$\underline{E} = -R, \text{ and} \tag{137}$$
$$\underline{E}^* = -R\sqrt{1+\dot{R}^2(1-F)}. \tag{138}$$

Evaluating these expressions at $r = r_+$ is straightforward with the only complication being $\dot{R}_+ \equiv \dot{R}(r_+)$. We have,

$$\dot{R}_+ = \frac{1}{1+a^2}\left(\delta^{a^2} + \frac{a^2}{\delta}\right). \tag{139}$$

If $a^2 \ll \delta$ then the square of the coupling constant is extremely small even relative to δ, and $\dot{R}_+ \approx 1$. In fact even if $a^2 \approx \delta$ then $\dot{R}_+$ is of the same order as 1. By contrast for $a^2 \gg \delta$, $\dot{R}_+ \approx \frac{1}{1+a^2}\frac{a^2}{\delta} \gg 1$. Thus, we calculate the quasilocal energies for the cases $a^2 <\approx \delta$ (which includes the magnetic Reissner Nordström case for $a = 0$) and $a^2 \gg \delta$ separately. The results along with those for $r \to \infty$ are displayed in figure 2. Note that for $a^2 <\approx \delta < 1$, $R_+ \approx r_+$.

From figure 2 we see that static observers outside a naked black hole measure $E_{Geo}, E_{tot} \to 0$ near to the horizon while the infalling observers measure those same quantities to be very large. This effect occurs for both $\delta \ll a^2$ and the $a^2 <\approx \delta$ (which include the RN holes) and so cannot be attributed to the "nakedness" of the holes. Of course since we have omitted the reference terms, both of these expressions blow up if we take the quasilocal surface out to infinity.

If we include the reference terms, then near to the horizon $E_{tot} - \underline{E}$ is very large for static observers where it is R_+. It is even larger in the absolute sense for infalling observers who measure it as $-R_+/\delta$. Again however, those effects are seen by observers surrounding both naked and near-extreme RN holes and so cannot really be attributed to the curvature. As $r \to \infty$ the two expressions agree which is not surprising since as $r \to \infty$ the velocity of the infalling observers goes to zero. Note however that we do not obtain the ADM mass.

More interesting are the measurements of $E_{Geo} - \underline{E}$. If $a \approx 1$ and the holes are large ($R_+^2 \gg 1$) then we see that while static observers near to the horizon measure large values, sets of observers falling into naked black holes actually measure very small values for this quasilocal energy. By contrast observers falling into an RN hole will measure large values. In fact we can see that for $a \approx 1$ and $R_+^2 \gg 1$ then these infalling observers will measure $E^*_{Geo} - \underline{E}^* \ll 1$, if and only if the black hole is naked. Thus this is an alternate characterizing feature of naked black holes when the coupling constant is of a reasonable size. The equivalence is broken if $a^2 <\approx \delta$ in which case the static and infalling observers both measure large energies. Consider for example the case where $a^2 = \delta$. Then the black hole can still be naked if δ (and therefore a^2) is small enough that $\delta^2 R_+^2 \ll 1$.

4.2.2 Why do naked holes behave this way?

At the beginning of the previous section we suggested that the curvature results could be understood in terms of the rates of change of the surface area of shells of infalling observers. In this section we explore this idea if more detail and also use it to provide an explanation of the $E_{Geo} - \underline{E}$ result.

First we quantify the expectation that the surface area of a shell of infalling observers will be changing extremely quickly as they cross the horizon of a naked black hole. Recall that naked black holes are near extreme and so the singularity sits "just behind" the horizon ($r_- \approx r_+$). More rigorously, Horowitz and Ross [25] noted that an observer passing through the horizon after falling from r_0 (the situation described by equation (95)), will hit the singularity at r_- after a proper time of $\Delta s <\approx \frac{r_+ - r_-}{F(r_0)} = \frac{R_+ \delta}{F(r_0)}$. Thus a set of observers infalling on geodesics that were stationary at infinity ($F(r_0) = 1$) will only have a very short time before they reach r_-. At r_-, $R(r) \to 0$ and so the area of the shell goes to zero. However, by assumption $R_+ \gg 1$ so at the horizon itself, that same area is very large. For the area to go from very large to zero in such a small time, we would naively expect it to be decreasing very quickly as the observers pass the horizon. To quantify this we use (56) to show that the fractional rate of change of the area of the surface Ω_t as measured by the observers who inhabit that surface is

$$\frac{\dot{A}}{A} = \frac{8\pi \int_{\Omega_t} d^2x \sqrt{\sigma} \varepsilon^{\updownarrow}}{\int_{\Omega_t} d^2x \sqrt{\sigma}} = -\frac{2\dot{R}_+}{R_+} = -\frac{2}{(1+a^2)R_+} \left(\delta^{a^2} + \frac{a^2}{\delta^2} \right), \tag{140}$$

If $a^2 \gg \delta$ (that is it isn't pathologically small), $\frac{\dot{A}}{A} \approx \frac{1}{R_+ \delta^2} \gg 1$ as we would expect. By contrast for the RN case ($a = 0$), $\frac{\dot{A}}{A} \approx \frac{1}{R_+} \ll 1$ However, our expectations are confounded if $\delta >\approx a^2 \neq 0$ in which case the hole remains naked even while the rate of change is more along the lines of the RN values. In that case the extremely small value of a suppresses the rapid decrease in area until the observers get even closer to the singularity (basically $r - r_- \ll a^2$).

These rates of change of the area also nicely explain why $E_{Geo} - \underline{E}$ is small while the observed curvature components are large. Recall that to define the reference term $\underline{E}$ we had to embed Ω_t into flat space along with a vector field $\underline{T}^\alpha$ defined so that if we evolved Ω_t with that vector field and made only intrinsic observations on the resulting timelike three surface then observers could not tell whether they were in the original or reference spacetimes. In particular, the area of Ω_t should change at the same rate. Thus, if the area decreases extremely rapidly, the embedded shell of observers in the reference spacetime would have to be moving at a correspondingly fast speed. Equation (131) quantifies this telling us that $\underline{v}_\vdash = \dot{R}/\sqrt{1 + \dot{R}^2}$ at the horizon. Then for $a^2 \approx 1$, $\dot{R} \gg 1 \Rightarrow \underline{v}_\vdash \approx 1$ and the observers have to move at close to the speed of light in the reference time to match the rate of change of the area. By contrast, for $a^2 \approx 0$, $\dot{R} \approx 1 \Rightarrow \underline{v}_\vdash \approx \frac{1}{2}$. The area is changing at a relatively leisurely rate so the observers do not need to move so fast in the reference time.

For observers moving at extremely rapid velocities there is a sense in which the relativistic effects of their speed become more important than those due to gravity. To see this recall equation

(3.4) in which we saw that $\varepsilon^2 - \varepsilon^{\updownarrow 2}$ is a constant independent of the speed of the observers. Now, by construction $\tilde{\varepsilon}^{\updownarrow}$ is the same in both the reference and original spacetime and so we can rewrite,

$$E^*_{Geo} - \underline{E}^* = \int_{\Omega_t} d^2x\sqrt{\sigma}\left(\sqrt{\underline{\varepsilon}^2 + \tilde{\varepsilon}^{\updownarrow 2}} - \sqrt{\varepsilon^2 + \tilde{\varepsilon}^{\updownarrow 2}}\right). \tag{141}$$

If $\tilde{\varepsilon}^{\updownarrow}$ is much larger than ε and $\underline{\varepsilon}$, then at the horizon

$$E^*_{Geo} - \underline{E}^* \approx \frac{1}{2}\int_{\Omega_t} d^2x\sqrt{\sigma}\left(\frac{\underline{\varepsilon}^2 - \varepsilon^2}{\tilde{\varepsilon}^{\updownarrow}}\right), \tag{142}$$

and so we see that as $\varepsilon^{\updownarrow}$ becomes larger and large the observers quasilocal energy becomes smaller and smaller. Physically, though ε^* and $\underline{\varepsilon}^*$ are boosted to be very large, the difference between them simultaneously becomes smaller and smaller. In particular for naked black holes we have

$$E^*_{Geo} - \underline{E}^* \approx 2\pi R_+^2\left(\frac{\underline{\varepsilon}^2}{\tilde{\varepsilon}^{\updownarrow}}\right) = \frac{R_+}{2\dot{R}_+} = -\frac{A}{\dot{A}}, \tag{143}$$

and we see that in this case the geometric quasilocal energy is actually the inverse of the rate of change of the area.

By contrast $E^*_{Tot} - \underline{E}^*$ includes matter terms which are also boosted to be very large. There is no corresponding term in the reference spacetime to cancel these large terms out. The result is the matter terms dominate over the geometrical terms in $E^*_{Tot} - \underline{E}^*$ and so this total quasilocal energy is very large.

5 Discussion

In order to investigate how different sets of observers would define energy, angular momentum and charge in systems of finite size, it is necessary to make use of a formalism which explicates the relationships between those observers. The non-orthogonal formalism developed in previous references [21, 23, 24] is designed to address this issue. We have in this paper extended it to include electromagnetic and dilatonic matter fields.

Our applications of this extended formalism have yielded a number of results. The QLE naturally breaks up into a gauge independent geometric part and a gauge dependent part. Likewise, the Hamiltonian only inherits a subset of the gauge invariance of the full action. These situations arise because of the manner in which the quasilocal formalism separates out the bulk and boundary terms of the action. For a particular gauge choice which is natural to make in stationary spacetimes, the gauge dependence of the Hamiltonian is eliminated and that of the QLE is reduced to that resulting from where we choose to set the zero of a the Coulomb potential.

We have also shown that the quasilocal energy, angular momentum, and spatial stress have the expected covariance properties under Lorentz-type transformations, allowing us to meaningfully relate observations made by static observers to those made by non-static ones. However this covariance is not respected by the reference background spacetimes that are used to render the various quasilocal quantities finite. The reason that the covariance is not respected is that the quasilocal quantities in the reference spacetime transform with respect to a different velocity than the velocity of the quasilocal surface in the original spacetime. The requirement that the two surface evolve in the same way in both spacetimes (which after all have different geometries) forces these two velocities to be different.

When we apply this formalism to various physical situations, we find a variety of results which both confirm and confound our intuition. The thin-shell formalism provides a natural operational definition of the QLE: the QLE contained within a 2-surface can be defined as the energy of a shell of matter that has the same intrinsic geometry as this 2-surface and a surface stress energy tensor defined so that the spacetime outside the 2-surface is identical to the original spacetime, while inside it is identical to the reference spacetime. A consideration of RN and naked black holes yields an interesting set of similarities and differences between the QLEs in each case as measured by differing sets of observers. For both types of black holes, the geometric energy E_{Geo} and total energy E_{tot} vanish for static observers near the event horizon, whereas infalling observers measure these same quantities to be very large. When the reference terms are included, both static and infalling observers measure $E_{tot} - \underline{E}$ to be very large for both kinds of holes. However measurements of $E_{Geo} - \underline{E}$ will differentiate between the two kinds of holes. Infalling observers will find this quantity to be large for RN holes but small for naked holes, provided that the coupling constant a is not too small.

Some outstanding issues remain. The inclusion of dyons remains an unsolved problem within our formalism. A natural first approach for dealing with such objects would be to consider different patches of the quasilocal surface, with different (but gauge-related) nonsingular gauge fields defined on each. Unfortunately this is not sufficient to solve the problem. A simple way of seeing this is to note that for a pure Maxwell spacetime $F^2 = 2(B^2 - E^2)$. With E^α and B^α not being treated on an equal footing in the action it is not so surprising that it they are not treated on an equal footing in the QLE defined from that action. Thus it appears that the action itself will have to be modified in some way to resolve this problem.

The relationship between the thin-shell formalism and QLE defined with AdS/CFT-inspired intrinsic reference terms also deserves further study. Although the correspondence between the two formalisms does not carry through, the intrinsic reference terms used so far have only been those which are the least divergent for quasilocal surfaces of large mean radius. Other intrinsic reference terms could be included for finite-sized quasilocal surfaces – their role in the thin-shell QLE correspondence remains to be explored.

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References

[1] J.D. Bekenstein, *Phys. Rev. D*$\mathbf{7}$, 2333 (1973).

[2] S.W. Hawking, *Nature* **248**, 30 (1974); *Comm. Math. Phys.*$\mathbf{43}$, 199 (1975).

[3] G.W. Gibbons and S.W. Hawking, *Phys. Rev. D*$\mathbf{15}$, 2752 (1977).

[4] For a review see A. Peet, Class. Quant.Grav. **15** 3291 (1998).

[5] A. Ashtekar, J. Baez, A. Corichi and K. Krasnov, Phys. Rev. Lett. **80** 904 (1998).

[6] J.D. Brown and J.W. York, *Phys. Rev. D*$\mathbf{47}$, 1407 (1993).

[7] J.D. Brown and J.W. York, *Phys. Rev. D*$\mathbf{47}$, 1420 (1993).

[8] J.D. Brown, J.D.E. Creighton, and R.B. Mann, *Phys. Rev. D*$\mathbf{50}$, 6394 (1994).

[9] J.D.E. Creighton and R.B. Mann, *Phys. Rev. D*$\mathbf{52}$, 4569 (1995); J.D.E Creighton, Ph. D. Thesis *Gravitational Calorimetry*, Department of Physics, University of Waterloo, Canada (1996).

[10] J.D. Brown, *Phys. Rev. D*$\mathbf{51}$, 5725 (1995).

[11] J.D. Brown, *Phys. Rev. D*$\mathbf{56}$, 1001 (1997).

[12] I.S. Booth and R.B. Mann, *Nucl. Phys. B*$\mathbf{539}$, 267 (1999); *Phys. Rev. Lett.* **81**, 5052 (1998).

[13] E.A. Martinez, *Phys. Rev. D*$\mathbf{50}$, 4920 (1994).

[14] E.A. Martinez, *Class. Quantum Grav.***13** 481, (1996).

[15] R. Arnowitt, S. Deser, C. Misner, *Gravitation: an introduction to current research* (Wiley, New York, 1962).

[16] A. Trautman, *Bull. Acad. Pol. Sci. Série des Sci. Math. Ast. Phys.* **6**, 407 (1958); H. Bondi, M.G.J. van der Burg, and A.W.K. Metzner, *Proc. R. Soc. London A* **269**, 21 (1962); R.K. Sachs, *Proc. R. Soc. London A* **270**, 103 (1962); R.K. Sachs, *Phys. Rev.* **128**, 2851 (1962).

[17] J.D. Brown, S.R. Lau, and J.W. York, *Phys. Rev. D*$\mathbf{55}$, 1977 (1997).

[18] S.W. Hawking and G.T. Horowitz, *Class. Quantum Grav.***13**, 1487 (1996).

[19] L. Abbot and S. Deser, *Nucl. Phys. B*$\mathbf{195}$, 76 (1982).

[20] S.Y. Hyun, W.T. Kim, and J. Lee, *Phys. Rev. D*$\mathbf{59}$, 084020 (1999); V. Balasubramanian and P. Kraus, hep-th 9902121; S.R. Lau gr-qc 9903038; R.B. Mann, hep-th 9903229; R. Emparan, C.V. Johnson, and R.C. Myers hep-th 9903238.

[21] G. Hayward *Phys. Rev. D*$\mathbf{47}$, 3275 (1993).

[22] S.R. Lau, *Class. Quantum Grav.***13**, 1509 (1996).

[23] S.W. Hawking and C.J. Hunter, *Class. Quantum Grav.***13**, 2735 (1996).

[24] I.S. Booth and R.B. Mann, *Phys. Rev. D*$\mathbf{59}$, 064010 (1999).

[25] G.T. Horowitz and S.F. Ross, *Phys. Rev. D*$\mathbf{56}$, 2180 (1997); *Phys. Rev. D*$\mathbf{57}$ 1098 (1998).

[26] H. Flanders, *Differential Forms with Applications to the Physical Sciences*, (Dover, 1989).

[27] S.W. Hawking and S.F. Ross, *Phys. Rev. D*$\mathbf{51}$, 5725 (1995).

[28] R.B. Mann and S.F. Ross, *Phys. Rev. D*$\mathbf{52}$, 2254 (1995).

[29] K.C.K. Chan, J.D.E. Creighton, and R.B. Mann, Phys. Rev. **D54** (1996) 3892.

[30] W. Israel, *Nuovo Cimento* **44B**, 1, (1966); erratum **48B**, 463 (1967).

[31] J.D. Jackson, *Classical electrodynamics*, (Wiley, 1975).

[32] M. Rakhmanov, Phys. Rev. **D50** 5155 (1994).

[33] P. Musgrave, D. Pollney, K. Lake, *GRTensor II, 1994, Queen's University, Kingston, Ontario, Canada.*